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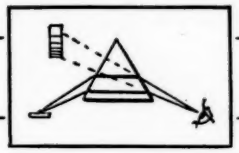

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Ultraviolet Wave-Length Standards of N, C, and O, λ2300 to λ1080

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(Received May 11, 1936)

Wave-lengths of N, C and O lines in the spectral range λ2317 to λ1080 have been determined with an accuracy of 0.01A to serve as wave-length standards in this region. A ten-foot focus normal incidence vacuum spectrograph having a dispersion of 5.5A/mm was used to compare second order N and C lines with first order Cu and Fe interferometer-measured standards. The results are compared with values given by Boyce and Rieke. The mean deviation from their values is 0.009A, the mean systematic deviation 0.001A. The use of the method of coincidences in determining tertiary standards is discussed.

THE increasing use of vacuum spectrographs creates a need for wave-length standards in the extreme ultraviolet region. The spectra obtained often contain lines due to impurities in the gas discharge light source. It is desirable that identification lists be available for the common impurities, H, He, C, N, O, Ne and A,[1] and that their wave-lengths be known with accuracy sufficient for wave-length standards.

The method of coincidences is at present the most feasible means of establishing provisional standards in the extreme ultraviolet. Doubtless other methods will be made available by the perfection of vacuum interferometers[2] and by the accumulation of sufficient data to make possible the use of the combination principle in calculating wave-lengths in the ultraviolet. Boyce and Rieke[3] have used the method of coincidences to determine wave-lengths standards of N, C, O and A. It is desirable that another observer check their results, for substantial agreement between measurements by several observers must be expected before lines may be used confidently as standards. Boyce and Rieke used a grating ruled at Johns Hopkins University. The work described in this paper was done with a grating ruled in 1932 at the University of Chicago.

To determine the standards listed in this paper, second order lines of N, C and O, excited in an enhanced discharge, were brought into direct comparison with first order Fe and Cu standards in the region 2350 to 2100. The ten-foot focus normal incidence vacuum spectrograph used gave a dispersion of 5.5A/mm. Interferometer determination of the Fe and Cu standards[4, 5, 6] were used in reducing the plates. Cu lines were found to be slightly preferable for this reduction. They happened to be influenced less by near-lying lines than were the Fe lines, and they gave

[1] J. C. Boyce and H. A. Robinson, J. Opt. Soc. Am. **26**, 133 (1936).
[2] W. E. Williams, Proc. Phys. Soc. (London) **45**, 699 (1933).
[3] J. C. Boyce and C. A. Rieke, Phys. Rev. **47**, 653 (1935).
[4] K. Burns and F. M. Walters, Jr., Pub. Allegheny Obs. VIII, No. 4, 39 (1931).
[5] K. Burns, Pub. Allegheny Obs. VI, No. 7, 159 (1929).
[6] K. Burns, Pub. Allegheny Obs. VIII, No. 3, 27 (1931).

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somewhat smoother plate calibration curves. Hence, although separate reductions (in good agreement) were made using Fe and Cu, the values of the wave-lengths listed in this paper are based largely upon Cu interferometer measurements.

Seven plates were obtained using He with N, C and O as impurities in the discharge tube. Cu and Fe comparisons were registered by focusing arcs on the slit through a quartz window in the rear of the discharge tube. The arcs were run at atmospheric pressure, with 10 mm electrode separation and currents of 4 to 5 amp.

Plates were reduced by first assuming a linear relation between wave-length and distance along the plate. By use of the Cu and Fe standards a plate calibration curve was obtained from which corrections to the approximate wave-lengths could be read. The curve was extended to lower wave-lengths by using as standards in this region six N lines whose wave-lengths had been determined by coincidence in the second order, and the H line of computed wave-length 1215.664A. Corrections to approximate wave-lengths read from the reduction curves were less than 0.05A in the region of overlapping standards, and less than 0.36A in the region of lower wave-lengths.

Before proceeding to the results of this investigation it may be well to consider briefly the types and magnitudes of the errors inherent in the work. Possible sources of systematic error are the error in the mean value of Cu and Fe comparisons, and the possible error arising from the method of coincidences. The first of these might occur if light from the sources failed to fill the grating or if temperature changes during the long gas exposures caused changes of dimensions. Observation of the grating illumination and thermostat temperature control are believed to have minimized these errors.

It is probable that the principal errors in the wave-lengths here presented are in the comparator measurements of the seven plates and in the drawing of calibration curves. A typical set of measurements of a single line on one plate gave a maximum deviation of 0.003 mm and a mean residual for ten settings of less than 0.002 mm or 0.011A.

A source of error not often considered in spectroscopic work is the irregular distortion of the photographic emulsion after developing and washing. Investigations by Schlesinger[7] and by D. Cooksey and C. D. Cooksey[8] place the magnitude of such local distortions at about 0.001 mm. These random effects may be minimized by averaging measurements from a number of plates.

In checking the wave-lengths obtained by Boyce and Rieke, it was necessary to know the Fe standards used by these authors. This information was kindly provided by Dr. C. A. Rieke and Dr. K. R. More. In the present work Cu and Fe arcs in air were used for comparisons although wave-lengths characteristic of vacuum arcs were used for plate reduction. This procedure is justified by the fact that the wave-lengths values used were the most accurate obtainable and that the pressure shift for one atmosphere is of the same order of magnitude as errors in measurement. The values of Fe standards used by Boyce and Rieke (after correction for the index of refraction of air by application of the Meggers and Peters formula[9]) were each 0.001A greater than those used in the present work. We infer that this discrepancy represents an arbitrary correction applied for a pressure shift. The mean systematic deviation in N and C wave-lengths, $\lambda_{B,\,R}-\lambda_{W,\,W}$, happens to be also just 0.001A. Thus there is no real systematic deviation if the difference in standards is considered.

In view of the opinion once expressed by Kayser[10] against the validity of the method of coincidences and the discordant work done by others[11] it may promote clarity to refer to the papers of Cornu[12] and Rydberg.[13] These deal analytically and experimentally with the effects of grating asymmetry and systematic variations in rulings. The results of the present work of themselves could not have given any positive arguments against the validity of the coincidence

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[7] F. Schlesinger, Pub. Allegheny Obs. I, No. 1, 1 (1910)
[8] D. Cooksey and C. D. Cooksey, Phys. Rev. **36**, 80 (1930).
[9] W. F. Meggers and C. G. Peters, Bull. Bur. Stand. **14**, 697 (1918–19).
[10] Kayser, Astrophys. J. **19**, 157 (1904); **32**, 217 (1910).
[11] J. T. Howell, Astrophys. J. **39**, 230 (1914).
[12] A. Cornu, Comptes Rendus Acad. Sc. (Paris) **80**, 645 (1875); **116**, 1215, 1421 (1893); **117**, 1032 (1893); **122**, 1455 (1896).
[13] J. R. Rydberg, Phil. Mag. [5] **35**, 190 (1893); Astron. and Astrophys. **12**, 439 (1893); **12**, 562 (1893).

TABLE I. *Wave-length standards from λ2300 to λ1080.* Asterisks indicate wave-lengths which have been measured in the second order directly against Fe and Cu.

Spectrum	Int.	Wt.	Vacuum λ	Boyce[3, 1] and Rieke
N II	1	17	2317.708	
N II	0	15	2317.219	
N II	2	19	2190.604	
C I	5	10	1930.889	.900
N I	7	19	1745.255	.246
N II	2	10	1743.226	.22
N I	7	19	1742.733	.734
N II	2	10	1740.320	.315
N II	2	5	1675.759	.83
C I	1b	8	1658.127	.126
C I	0	6	1657.879	.916
C I	2	6	1657.381	.380
C I	2	18	1561.414	.42
C I	2	16	1560.703	.699
C I	0	14	1560.323	.316
N I	5	18	1494.668	.669
N I	6b	12	1492.634	.630
N I	5	18	1411.935	.937
N II	1	3	1346.400	.41
N II	0	3	1345.311	.29
C II	4	13	1335.713	.71
C II	4	14	1334.541	.54
C I	1	10	1329.625	.58
C I	1	12	1329.092	.101
N I	0	5	1327.960	.956
N I	1b	11	1326.629	.610
N I	1	14	1310.967	.973
N I	1	14	1310.569	.566
O I	0	4	1304.850	.864
N II	2	12	1276.175	.18
N I	4	11	1243.141	.170
H[2]	3	21	1215.664 calc.	
N I	3	15	1200.693	.706
N I	3	16	1200.215	.220
N I	4	16	1199.557	.547
N I	0	9	1177.677	.673
N I	0	10	1176.498	.504
N I	3b	3	1168.477	.475
N I	0	11	1167.454	.424
N I	0	2	1163.858	.850
C I	3b	3	1139.885	.894
N I	3	21	*1134.980	.980
N I	3	21	*1134.416	.419
N I	3	21	*1134.169	.171
N II	2	19	*1085.726	.74
N II	2	19	*1085.546	.546
N II	2	19	*1084.582	.579
N II	2	12	*1083.990	.991

method. The fact that good agreement is obtained with the values of, Boyce and Rieke however, does give additional experimental indication that errors possibly inherent in the method do not exceed the estimated accidental errors of measurement. On the basis of this comparison of results obtained using gratings ruled on different engines, and on the basis of results obtained by other observers who have investigated the coincidence method, it is concluded that the method of overlapping orders can give tertiary standards of accuracy comparable with that obtained by direct interpolation between interferometer-measured standards.

Our results are given in Table I. The weights assigned are products of the number of plates by the number of measurements. Intensities are on an arbitrary scale. Asterisks indicate those lines which were measured in their second orders directly against Cu and Fe standards. There is close agreement for these wave-lengths with the Boyce and Rieke values. Over the entire region the mean deviation from their values is 0.009A, the mean systematic deviation 0.001A.

We wish to thank Professor H. S. Uhler for helpful discussion of the method of coincidences.

Photographic Effectiveness of the Radiation from a Number of Photographic Sources

W. E. FORSYTHE AND MISS M. A. EASLEY, *Incandescent Lamp Department, General Electric Company, Nela Park, Cleveland, Ohio*

(Received May 22, 1936)

THE source of light is very important in photography, and its value for different purposes depends upon both the intensity and the quality of its light. The standard source for sensitometry has the same quality as the noonday sun, and is obtained by filtering the light of a tungsten lamp operated at a definite temperature through a particular filter.[1] Just why is it that the intensity and the character of the radiation are so important, and why is it necessary to produce special lamps of high intensity for photographic work? Fundamentally, of course, this can be blamed on the photographic plates or films, since they require a definite amount of radiation of a particular quality to fall upon the plate or film in a very definite manner to produce the necessary chemical changes to record the impression. While the actual amount of radiation needed is small, the means for getting the necessary amount of radiation to the plate are all very wasteful of light.

Needs for artificial sources vary all the way from very intense sources for taking commercial moving pictures to smaller sources for taking snapshots or pictures in the home. For a long time photographic studios have been using the regular 500 w and 200 w incandescent lamps as light sources for illuminating their subjects. For special motion picture work 1000 w, 1500 w, 5000 w and, at times, even 10,000 w incandescent lamps have been used. These lamps have within a few percent the same photographic effectiveness as the movie flood lamp, lumen for lumen, since they are operated at about the same temperature. In addition, two types of lamps, the photoflash and the photoflood, have been developed as special sources for photography. The photoflash lamp,[2] as the name implies, is a lamp intended for taking flashlight pictures. This requires an intense flash of light which is produced in the lamp by the burning of very thin aluminum foil in the presence of an excess of oxygen. The bulb of the flash lamp contains pure oxygen at a pressure of about 180 mm of mercury and aluminum foil about 0.00004 cm thick in amounts that depend upon the size of the bulb. To start the foil burning, the filament of the lamp is covered with a primer that ignites when the filament is heated by an electric current. This lamp is now manufactured in three sizes with characteristics as shown in Table I.

TABLE I. *Characteristics of photoflash lamps.*

Lamp No.	Diameter of Bulb (cm)	Weight of Aluminum (mg)	Time from Turning on Current to Start of Flash (sec.)	Duration of Flash (sec.)	Total Light Output (lumen seconds)	Maximum Intensity (million lumens)
10	6.0	30	0.010	0.050	22,000	2.0
20	7.3	60	.008	.060	40,000	4.0
75	10.2	240	.017	.100	180,000	10.0

The time intensity curves for the three different sizes of photoflash lamps are shown in Fig. 1. The spectrum of the radiation from a photoflash lamp shows a few bright lines, a number of bands, and an intense continuous

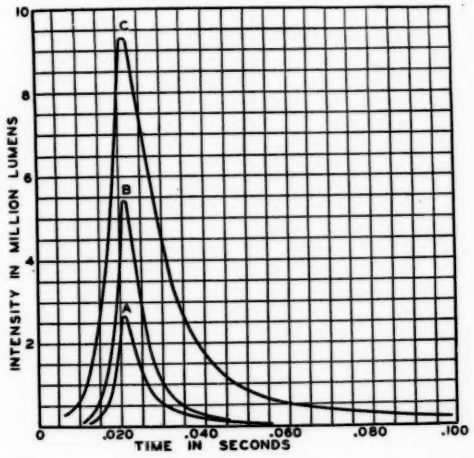

FIG. 1. Time intensity curves for three different types of photoflash lamps. The curve for the No. 75 lamp has been shifted 0.021 second toward origin. *A* = No. 10, *B* = No. 20, *C* = No. 75.

[1] Davis and Gibson, Nat. Bur. Stand. Misc. Pub. No. 114 (1931).

[2] J. O. S. A. **24**, 195 (1934).

TABLE II. *Characteristics of photoflood lamps.*

Lamp	Watts	Life (hrs.)	Light Output (lumens)	Color Temperature (°K)
No. 1 photoflood	250	2	8,600	3490
No. 4 photoflood	1000	10	32,000	3410
Movieflood	2000	15	67,000	3430
Regular 115 v	500	1000	9,800	2940

background. The spectral distribution of the radiation from the No. 20 photoflash lamp is shown in Fig. 2. The spectral distribution for the No. 10 lamp is probably quite similar to this, the radiation being, if anything, relatively more intense in the blue part of the spectrum, while the radiation from the No. 75 lamp is probably less intense in the blue part of the spectrum.

The photoflash lamp suggested that it would be desirable to have an incandescent lamp of small size and high intensity for taking pictures in the home. This idea resulted in the development of two photoflood lamps and a movie flood lamp with characteristics as shown in Table II. By making these lamps for a short life, a very great increase in the light output for the same wattage and also an increase in the photographic effectiveness, lumen for lumen, of about 25 percent for panchromatic films, as compared to the regular lamps of 1000 hour life, is obtained. This is due to the higher temperature of operation of the filament. The distribution in the visible spectrum of these photoflood and movie flood lamps is not markedly different from that of the No. 20 photoflash lamp which is shown by the curves of Fig. 1.

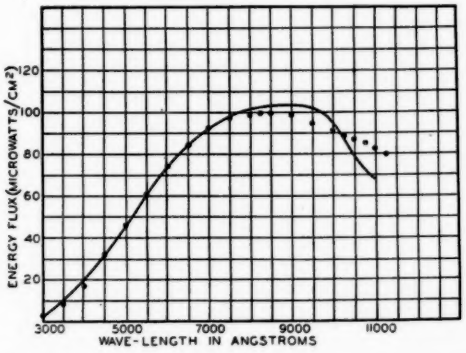

FIG. 2. Spectral distribution of radiation from No. 20 photoflash lamp. The points give the energy radiated by a black body at 3500°K.

The early photographic plates or films were sensitive only to the blue part of the visible spectrum and to the long ultraviolet radiation, but now orthochromatic and panchromatic emulsions are made that are sensitive in the yellow, and, in the case of the panchromatic, far into the red end of the spectrum. These different photographic materials are blackened differently by the light from any one source, and each reacts differently to radiations of different spectral characteristics. The ability of radiation of different sources to produce blackening on a photographic plate or film has been designated as the photographic effectiveness of the radiation, a quantity, of course, varying with the characteristics of the photographic material used.

To study the relative photographic effectiveness of the photoflood and the photoflash lamps, the regular 115 v, 500 w incandescent lamp was used as the comparison standard and comparable densities of the films were obtained by varying the distance between the lamp and the film. The films studied were mounted on the drum of the rotating flashometer.[3] This drum was provided with an arrangement for exposing in turn small strips of the film, through an accurately adjusted slit, to the light of the lamp studied. The film was calibrated by varying the distance between the 500 w lamp and the film, and the distance from the film necessary to operate the test lamps to produce comparable densities was obtained by trial or by computation. The relative photographic effectiveness of the radiation of photoflash lamps at the time of maximum output and the 500 w lamp for calibration was obtained from films exposed with the drum rotated at a constant speed. To obtain the relative photographic effectiveness of total radiation from the photoflash lamps and the incandescent lamps, the slit was removed from the front of the flashometer, thus exposing round spots (about 8 mm in diameter) to the radiation with the drum stationary, the length of exposure for the incandescent lamps (about 1/25 second) being determined by a calibrated shutter.

Since the total light output of the larger photoflash lamps is so great and hence would result in a very dense spot on the film for any distances available in the laboratory, and since

[3] R. S. I. **5**, 216 (1934).

TABLE III. *Relative photographic effectiveness of some photographic lamps for different types of plates.*

Lamp	Watts	Type of Emulsion				
		Process	Commercial	Orthochromatic	Portrait Panchromatic	SS Panchromatic
Regular 115 v	500	1.0	1.0	1.0	1.0	1.0
No. 1 photoflood	250	1.6	1.3	1.2	1.1	1.1
No. 4 photoflood	1000	5.3	4.6	4.5	4.0	4.2
Movieflood	2000	12.0	11.2	10.5	8.8	9.3
Maximum light intensity						
No. 10 photoflash		510	450	430	310	350
No. 20 photoflash		850	840	690	520	460
No. 75 photoflash			2200	1650	1250	1290
Total light output						
No. 10 photoflash		190	190	135	100	90
No. 20 photoflash		270	270	210	170	170
No. 75 photoflash		1460	1270	860	790	760

there is such a great difference between the total light output of the photoflash lamp and the 500 w incandescent lamp for 1/25 second, it was necessary to adopt some means to reduce the intensity of the radiation that fell upon the film from the photoflash lamp, in order to be able to get measurable density for comparisons for the distances available in the laboratory. This was done by mounting a tube about 60 cm long in front of the flashometer and covering the outer end of the tube with a screen made of two pieces of very clear glass which had been etched on both sides. Tests showed that such a screen approximated closely the distribution given by Lambert's cosine law. This method of reducing the intensity is very nearly non-selective, and, as used, gave sufficient reduction so that satisfactory densities were obtained even with the largest photoflash lamp at a distance of no more than 18 meters from the lamp to the screen. The film was calibrated with the screen in place and the 500 w incandescent lamp near enough to the screen to obtain comparable results and still have the photographic film effectively "see" the entire light source of the lamp.

In all, five different types of films from two different manufacturers were tested. All films were developed in D-76 developer to a contrast of 0.8. Curves were plotted to give the relationship between the density of the trace and the energy falling upon the film, and the relative energy required to produce a density of 1.0 was then determined from these curves. Table III gives the data on relative photographic effectiveness found for the lamps tested under the stated conditions.

Spectral Radiant Intensities of Some Tungsten Filament Incandescent Lamps

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(Received June 3, 1936)

THE spectral distribution of the energy radiated by a tungsten filament lamp is determined to a large extent by the temperature of the filament. The normal temperature of operation of the filament of a tungsten lamp depends on the diameter of the wire and the way in which it is coiled and mounted, the gas used in the lamp, and the life for which it is designed. Thus, the 500-watt lamp for general lighting purposes has a filament that operates at a temperature of 2940°K, while that of the 40-watt lamp operates at a temperature of about 2710°K. Both of these are designed for 1000 hours life. On the other hand, while the filament of the 500-watt lamp for general lighting purposes, designed for 1000 hours life, operates at a temperature of about 2940°K, the filament of the 500-watt CX lamp operates at a temperature of about 3025°K for a life of about 500 hours, and the monoplane filament of the 500-watt lamp for projecting moving pictures operates at a temperature of about 3250°K and has a life of about 100 hours. These differences in temperature of operation adapt these lamps to their different uses. In projecting moving pictures, efficient production of light is much more important than a long life; hence, the amount of light radiated is increased by using a filament which operates at a higher temperature than normal for lamps for general use. The life of the lamp is, of course, shorter, as the useful life of a tungsten lamp decreases very rapidly with increase in temperature.

The spectral radiant energy of wave-lengths between 3500 and 26000A was measured for a number of representative lamps. For these measurements, the quartz double monochromator previously described[1] was used. The measuring device for the infrared and part of the visible was a thermopile; for the rest of the spectrum, it was a photoelectric tube. Some of the operating characteristics of these lamps are shown in Table I, and the spectral intensities are shown in Tables II and III. The values given in Tables II and III are the microwatts that fall upon a square centimeter one meter from the lamp for a 50A wave-length interval for the wave-length indicated. In the case of lamps having the filament in one plane, the energy radiated perpendicular to the plane of the filament was measured; for the other lamps, the radiation perpendicular to the plane of the leads was measured with the leads turned away from, and the center of the filament turned toward, the spectrometer. The relative spectral distribution would probably be very nearly the same in any direction, but absolute magnitudes of the spectral intensities would vary with direction in the same ratio as the luminous outputs.

The spectral radiant intensities from a tungsten ribbon filament as transmitted through a hard glass bulb one mm thick are given for a number of temperatures and for a range of wave-lengths in Table IV.

The spectral distribution within the visible spectrum can be calculated from the color temperature[2] of the filament of the lamp. The

TABLE I. *Operating characteristics of a number of lamps.*

Lamp (watts)	Lumens per watt	Color Temp. (°K)	Luminous Intensity[1] (candles)
40	10.7	2760	30
100	15.3	2865	93
500	19.6	2960	590
1000	20.7	2990	1700
1500	22.0	3225	3200
900 Projection 30V–30A	26.5	3200	2700
500 Projection	26.3	3190	1500
500 CX	21.1	3015	800
250 CX	19.0	2950	370
60 CX	13.7	2825	56
6V, 21 cp. auto headlight		3015	23
Photocell exciter[2]			
7.5V		3055	560
6.0V		2640	140
Photoflood			
250W	35.8	3475	2300
1000W	31.6	3360	5500

[1] This is the horizontal candle power in the direction in which the spectral intensities were measured.
[2] This lamp has a horizontal coil of about 12 turns, and is in a tubular bulb about 1 inch in diameter and 2 inches long.

[1] Forsythe and Barnes, R. S. I. **4**, 289 (1933).

[2] Forsythe, J. O. S. A. **7**, 1115 (1923).

TABLE II. *Spectral radiant intensities*[1] *for a number of 115-volt lamps for general lighting purposes.*

Wavelength (Å)	Lamp Wattage 1500	1000	500	100	40
3500	1.15	0.75	0.225	0.030	0.0075
4000	4.05	2.22	0.700	0.098	0.0245
4500	9.0	5.0	1.55	0.238	0.067
5000	15.5	8.7	2.65	0.435	0.127
5500	22.2	12.7	3.95	0.66	0.200
6000	29.7	16.7	5.5	0.89	0.260
6500	36.0	20.1	6.5	1.12	0.355
7000	41.5	21.6	7.6	1.31	0.43
7500	45.5	25.5	8.5	1.47	0.48
8000	48.5	27.5	9.2	1.60	0.52
8500	50.5	28.8	9.6	1.70	0.56
9000	51.5	29.4	10.0	1.75	0.58
9500	52.0	29.2	10.2	1.80	0.60
10000	51.8	29.0	10.5	1.81	0.62
10500	51.0	28.7	10.1	1.82	0.62
11000	49.8	28.3	9.9	1.78	0.61
12000	45.7	26.5	9.3	1.69	0.60
13000	41.5	24.0	8.6	1.57	0.55
14000	36.8	21.2	7.7	1.43	0.50
15000	31.8	18.8	6.7	1.28	0.45
16000	27.5	16.5	5.8	1.14	0.40
17000	23.9	14.2	5.0	1.00	0.35
18000	20.8	12.5	4.7	0.87	0.31
19000	17.9	10.8	3.7	0.76	0.27
20000	15.5	9.5	3.3	0.66	0.24
22000	11.6	7.3	2.6	0.52	0.19
24000	8.7	5.5	2.1	0.39	0.15
26000	6.5	4.4	1.8	0.31	0.12

[1] Microwatts per square centimeter at one meter per 50A.

TABLE III. *Spectral radiant intensities*[1] *for a number of lamps for special purposes.*

Wavelength (A) \ Lamp Watts	900[2] Movie	500[2] Projection	500[3] CX	250[3] CX	60[3] CX	6.4 v[7] auto headlight	10 v[4] Photocell Exciter 7.5[5]	6.0[6]	Photoflood 250[3]	1000[3]
3500	1.01	0.56	0.35	0.16	0.019	.00104	0.065	0.008	0.45	1.90
4000	4.1	2.25	1.00	.48	.062	.00301	.200	.024	1.05	4.90
4500	8.2	4.7	2.20	1.04	.145	.0062	.405	.071	1.75	9.30
5000	13.5	7.5	3.73	1.76	.248	.0105	.67	.140	3.05	14.3
5500	18.8	10.7	5.4	2.52	.41	.0157	.99	.235	3.90	19.3
6000	23.5	13.4	7.1	3.58	.56	.0206	1.30	.355	4.60	23.0
6500	27.3	15.6	8.5	4.44	.69	.0257	1.57	.46	5.10	25.8
7000	30.4	17.5	9.7	5.07	.81	.0301	1.87	.56	5.50	28.1
7500	32.8	18.6	10.7	5.55	.90	.0332	1.90	.66	5.70	29.4
8000	34.4	19.5	11.5	5.95	.99	.0357	2.00	.74	5.75	29.8
8500	35.2	19.9	12.0	6.25	1.04	.0374	2.05	.80	5.70	29.9
9000	35.5	20.0	12.3	6.42	1.08	.0380	2.08	.86	5.60	29.6
9500	35.3	19.9	12.4	6.50	1.11	.0382	2.07	.90	5.40	29.0
10000	34.6	19.5	12.4	6.52	1.13	.0381	2.06	.92	5.20	28.2
10500	33.7	19.0	12.2	6.52	1.13	.0380	2.02	.94	5.00	27.0
11000	32.3	18.5	11.8	6.44	1.12	.0375	1.95	.94	4.70	25.7
12000	30.0	17.0	11.0	6.12	1.04	.0355	1.80	.91	4.20	23.1
13000	27.2	15.4	9.9	5.45	.98	.0350	1.65	.85	3.75	20.5
14000	24.0	13.6	9.0	4.85	.90	.0292	1.45	.79	3.25	18.1
15000	21.0	11.9	7.8	4.33	.80	.0258	1.30	.71	2.80	15.8
16000	18.3	10.2	6.8	3.62	.71	.0222	1.14	.63	2.40	13.6
17000	16.1	8.4	5.7	3.15	.62	.0191	1.00	.57	2.05	11.6
18000	13.8	7.6	5.1	2.74	.54	.0166	.87	.51	1.70	9.9
19000	12.0	6.4	4.3	2.39	.47	.0144	.76	.45	1.50	8.4
20000	10.5	5.5	3.8	2.09	.40	.0126	.67	.40	1.30	7.2
22000	8.7	4.2	2.9	1.63	.31	.0098	.53	.31	1.00	5.6
24000	6.4	3.4	2.2	1.26	.24	.0078	.39	.25	.75	4.6
26000	5.1	2.8	1.7	1.04	.20	.0066	.30	.19		

[1] The values given are the microwatts per 50A that fall upon one square centimeter at a distance of one meter from the lamp.
[2] 30-volt, 30-ampere lamp.
[3] 115-volt lamp.
[4] 10-volt, 7½ ampere lamp.
[5] Operated at 7.5 amperes—see note 2, Table I.
[6] Operated at 6.0 amperes.
[7] 21-candlepower lamp.

distribution thus found agreed within the experimental error with the measured values. The luminous intensity calculated from the spectral intensity values also agreed within the experimental error with the values measured by visual photometry. As a check, the spectral intensity in the visible spectrum of three high intensity incandescent lamps (30V, 30A movie lamps, operated at 25A) that had been standardized for horizontal luminous intensity by the National Bureau of Standards was measured with the same apparatus. The luminous intensity obtained from the spectral intensities agreed with the Bureau's value to within 2 percent.

It cannot be said that the spectral radiation of the different lamps is that of tungsten at the temperature of operation of the filament for these reasons: the absorption of the glass bulb very markedly changes the distribution for wavelengths greater than about 2.5μ; the filament is not all at the same temperatures, due to end cooling, etc., which means that the measured distribution is approximately that for some average temperature; and the inter-reflection

TABLE IV. *Spectral steradiance of tungsten ribbon as observed through 702P glass bulb one mm thick.* Values as given are watts per square centimeter of radiating surface per unit solid angle.

Wavelength \ Temp.	3300°K	3000°K	2800°K	2700°K	2600°K	2150°K
3500	.049	.0145	.0054	.0032	.0017	.000066
4000	.142	.0490	.0216	.0128	.0077	.00045
4500	.261	.1035	.0466	.0309	.0196	.00161
5000	.402	.172	.0858	.059	.0394	.00417
5500	.537	.248	.133	.0946	.0654	.0085
6000	.660	.326	.184	.1345	.096	.0148
6500	.756	.393	.234	.1755	.1285	.0227
7000	.825	.447	.279	.212	.160	.0317
7500	.867	.487	.302	.245	.188	.0414
8000	.889	.516	.344	.273	.211	.0509
8500	.887	.531	.364	.291	.230	.0596
9000	.880	.545	.377	.306	.244	.0682
9500	.854	.524	.381	.312	.252	.0748
10000	.818	.512	.379	.3125	.257	.0804
10500	.778	.495	.372	.310	.256	.0834
11000	.736	.471	.361	.304	.253	.0869
12000	.648	.419	.334	.283	.238	.0889
13000	.564	.370	.302	.258	.218	.0870
14000	.489	.321	.267	.230	.195	.0826
15000	.418	.278	.234	.201	.173	.0763
16000	.365	.242	.203	.176	.153	.0742
17000	.303	.222	.176	.152	.133	.0611
18000	.259	.183	.149	.130	.114	.0556
19000	.222	.159	.1255	.112	.0988	.0496
20000	.190	.136	.1115	.097	.0854	.0433
22000	.136	.101	.0846	.074	.0648	.0337
24000	.102	.069	.0640	.0564	.0496	.0260
26000	.0760	.058	.0475	.0427	.0380	.0201
30000	.0275	.0235	.0196	.0178	.0161	.0088
35000	.0119	.0107	.0091	.0084	.0077	.0041
40000	.0030	.0025	.0021	.0019	.0017	.0010

between adjacent coils makes the radiation that comes from between the coils more nearly like that from a black body at the same temperature as the coils. Thus the radiation of the lamp is made up in part of radiation characteristic of tungsten, and in part of radiation with a spectral distribution somewhere between that of tungsten and that of a black body at the temperature of the filament.

It is somewhat difficult to estimate the accuracy of these results. However, it is thought that the values of the spectral radiation in the central portion of the spectrum will at least be reproducible to plus or minus 3 percent, while those at the ends of the spectrum are much less accurate.

The spectral emissivity of tungsten for wave-lengths greater than 5μ is small but appreciable (15 percent for a temperature of 2900°K and a wave-length range between 5 and 6μ), but the lamp bulb is practically opaque to radiation of these wave-lengths. Thus, for wave-lengths longer than 5μ, the radiation is that of hot glass. The amount of this radiation varies because the bulbs of the different lamps operate at different temperatures. The bulb must be at such a temperature that it can get rid of the energy it receives from the gas within the bulb and also of the direct filament radiation which it absorbs. For most lamps, the absorbed radiation amounts to about 5 percent of the input. The amount of energy that the gas carries to the bulb varies from about 25 percent of the input for the 40-watt lamp to about 7.5 percent for the 500-watt lamp. The bulbs of lamps for general lighting purposes operate at temperatures between 150 and 200°C, and the bulbs of some of the special lamps operate at temperatures of 300 to 400°C.

The Fluorescence of Rhodamine

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THE present experiments are concerned with the fluorescence characteristics of rhodamine films in solid solution, more particularly with the quantum yield and with the fading under continued exposure. Although many studies have been made of its fluorescence, there is no satisfactory information available on these two points.

The fluorescence itself is presumably a consequence of excitation of the atom, but the mechanism is more involved than in the case of gaseous luminescence. In a solid substance, the energy levels of the electron do not fall into sharply defined states as in a gas, but rather into broad bands separated by forbidden zones. The width of the latter depends on the lattice spacings and homogeneity of the crystal. Its magnitude, together with the energy distribution of the valence electrons, determines the possibility of activating the substance by absorption of radiation, and of causing thereby the ejection of electrons to higher energy bands. Their return gives rise to the fluorescence. The presence of a foreign substance, such as an appropriate solvent, may serve to introduce supplementary bands within the forbidden zone. These lower the energy required for the electron jumps and thus serve as stepping stones to the attainment of activation. The role of the solvent is an important one. The demonstration has been made by Schmidt[1] that certain organic compounds always regarded as non-fluorescent will nevertheless develop fluorescence under choice of a proper solvent. Similarly for rhodamine, as will be shown later, the extent or even existence of fluorescence depends upon the solvent used. The pure dry salt shows no fluorescence whatever.

The spectral distribution and energy content of the fluorescence were measured with a Bausch and Lomb spectrophotometer. Excitation was limited to the visible range by exposing samples to the radiation from a glass 250-watt, 220-volt, high intensity, mercury lamp, made by the General Electric Vapor Lamp Company. The relative intensity of the fluorescent energy throughout its spectrum was determined by comparison with a 200-watt Mazda incandescent lamp at 18 l.p.w. A slit width was used which corresponded to a range in wave-length of 20A at 5890A. The energy content was plotted for variation in wave-length, and the energy of fluorescence was taken as the area under this curve. The corresponding value for the absorbed radiation was found by measuring the energy content of the three prominent mercury lines 5461, 5770 and 5791A, first as reflected from a magnesia block and then as reflected from a film of rhodamine which had been rubbed with emery to give it a similar matte surface. The difference was a measure of the absorption.

The light from the high intensity lamp comprises a small amount of continuous radiation as a consequence of its operation at one atmosphere pressure of mercury. This involved a small correction to the fluorescent energy in order that the latter might represent only that produced under excitation by the lines 5461, 5770, and 5791A. By the use of suitable filters it was found that virtually all of the fluorescence was excited by these three mercury lines. That excited by 4358A amounted to less than 4 percent of the total. This is in accord with the results of Poritski,[2] who found the absorption in alcohol solution to show an optimum at 5500A, and to fall to very low values at 4358A, the spectral field covered by it remaining unaltered with variation in concentration. The absorption in solid films, according to Dixon,[3] is embraced within the same limiting wave-lengths as that for liquid solutions. Furthermore, the position and distribution of the fluorescent spectrum are not altered as the wave-length of the exciting light is varied, according to Wawilow,[4] and the total energy of fluorescence is proportional to the absorbed energy of the exciting line.

[1] Schmidt, Ann. d. Physik **65**, 247 (1921).

[2] Poritski, J. Frank. Inst. **197**, 527 (1924).

[3] Dixon, J. O. S. A. **21**, 250 (1931).

[4] Wawilow, Phil. Mag. **43**, 307 (1922).

Fluorescence Yield

The efficiency of conversion of the radiant energy of the mercury spectrum into fluorescent energy can be most conveniently expressed as the quantum yield, calculated on the basis of $E=h\nu$. Inasmuch as the fluorescent energy covers a considerable range, its effective wave-length was taken as that value which bisected into two equal parts the area covered by its spectral energy distribution. The yield is then expressed as the number of quanta of energy, absorbed from the lines 5461, 5770 and 5791A, necessary to form one quantum of fluorescent energy. The yield is also expressed in another way which is sometimes more useful, namely, the percentage which the fluorescent energy bears to the radiant visible energy of the lamp. These two quantities do not always show the same relationship, for the absorption factor is absent in the second.

Tests were made with a characteristic salt, rhodamine B, Schultz No. 573, having the empiric formula $C_{28}H_{31}N_2O_3Cl$ and the molecular weight of 478.8. Films were prepared by applying the solution in an appropriate solvent to a piece of matte white paper of uniform size pasted to a wooden block for convenience and accuracy of measurement. Satisfactory fluorescence could also be obtained from films painted on a glazed porcelain surface or on an aluminum surface roughened by rubbing with coarse emery. A glossy surface, such as that from polished aluminum, yielded scarcely any fluorescence due to the phenomenon of multiple reflection and consequent imprisonment of radiation, discussed by Wood.[5]

The magnitude of the yield is affected by several variable factors. An outstanding one is the concentration. This has been studied in the past especially for the case of solutions in alcohol. For very low concentrations the yield of fluorescence, (ϕ_o), per unit quantity of absorbed radiation remained constant, but at concentrations above a limiting value it fell off abruptly, as represented by the equation established by J. and F. Perrin,[6]

$$\phi=\phi_o e^{-kc},$$

in which c represents the concentration and k depends upon the character of the dye and of the solvent used. The situation was studied in more detail by Wawilow.[7] Recently it has been found by Dixon,[8] that dried collodion films containing rhodamine showed a decrease in fluorescence with concentration that likewise followed Perrin's equation. This decrease in yield has been ascribed to a disturbance of the emission process through some kind of resonance action. As the concentration increases, the probability of collision between dye molecules increases also, and with it the opportunity for the energy of an activated molecule to be dissipated into other forms rather than to be converted into light.

For a proper evaluation of the efficiency of conversion, it was important to find where the limiting concentration lay beyond which the yield of fluorescence decreased. The results are shown in Table I for films of constant thickness. Here, as likewise in all later tables, the concentration is taken as that of rhodamine in the liquid solvent. In this case the solvent was a 30 percent solution of cellulose acetate in acetone.

The upper limit of the absorption is at about 5900A. As the concentration increases, the fluorescent radiation of shorter wave-length is increasingly absorbed within the film. Consequently, the fluorescence actually emitted becomes lower in this range and relatively richer in the red. This accounts for the shift in the peak of fluorescence toward longer wave-lengths that accompanies an increase in concentration, as brought out in Table I and shown graphically in the curves of Fig. 1.

The concentration of $4(10)^{-4}$ molar is shown by Table I to be the limit beyond which the fluorescence decreases in accordance with Perrin's law. As it proved also to give the highest fluorescence yield, it was taken as the standard con-

Table I. *Relation of fluorescence yield to concentration.*

Molar Conc.	Peak λ	Fluor. as % of Radiant Energy	Quantum Yield
$4(10)^{-6}$	5900A	5.4	4.3
$4(10)^{-5}$	5930	28.3	2.0
$4(10)^{-4}$	6050	44.0	1.6
$4(10)^{-3}$	6210	4.6	12.5
$3(10)^{-2}$	6250	1.0	53.6

[5] Wood, Phil. Mag. **25**, 449 (1913).
[6] J. and F. Perrin, Comptes rendus **177**, 469 (1923); **178**, 2252 (1924).
[7] Wawilow, Zeits. f. Physik **31**, 750 (1925).
[8] Dixon, J. O. S. A. **21**, 250 (1931).

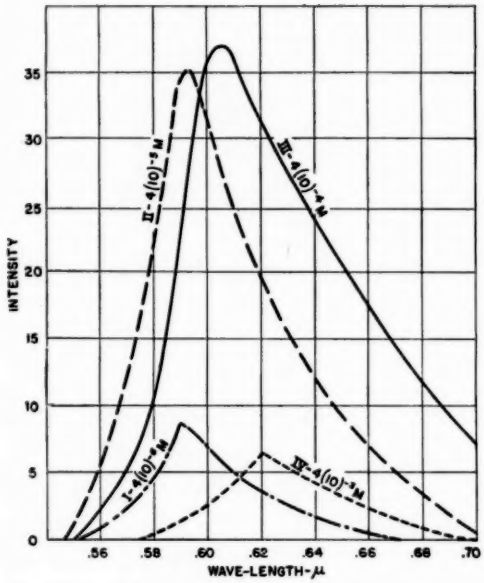


FIG. 1. Effect of rhodamine concentration upon fluorescence under constant excitation.

centration for all further tests. It corresponds to 0.2 gram of rhodamine B per liter of solvent.

The nature of the solvent is also important. In the work of others, it has not only given rise to variations in fluorescence intensity, but for certain dyes the very development of fluorescence itself depended upon the proper choice of solvent. Solid solutions have been most actively studied by Schmidt[9] and by Kautsky, Hirsch and Baumeister.[10] The present tests were made with available plastics, results from the best of which are recorded in Table II. Of many others tried, both organic and inorganic, the fluorescence was poor or absent. All films were of the same thickness and were prepared by the drying of liquid solutions of the standard concentration.

Celluose acetate not only allows of the greatest yield but, as will be shown below, it offers the greatest resistance to fading. Its use was standardized for all further tests.

A third significant factor is the thickness of the film. Its importance lies more particularly in its effect upon the extent of fading, as will be discussed later, but it is likewise of consequence in regard to the fluorescence yield. The results are given in Table III from tests made at the standard concentration in cellulose acetate films.

The trend of the results is brought out by the curves of Fig. 2. Particularly in regard to the percentage which the fluorescence bears to the radiant energy, it is noticeable that the relationship encounters a turning point at a thickness of about 14.0 $(10)^{-3}$ cm. Up to this point it is found that the results can be expressed in the form of the equation

$$dF/dx = k(F_{\max} - F_x),$$

where F_x represents the fluorescence in terms of its percentage to the radiant energy for a film having a thickness x. $F_{\max}$ is the maximum fluorescence of 44.0 percent attained at a thickness of 14 $(10)^{-3}$ cm. The terms of the equation imply that the increase in fluorescence, up to the point at which saturation occurs, is dependent

TABLE II. *Relation of fluorescence yield to solvent plastic.*

Plastic	Peak λ	Fluor. as % of Radiant Energy	Quantum Yield
Cellulose Acetate	6050A	44.0	1.6
Nitro-cellulose	6060	42.8	1.7
Formvar	6170	41.9	1.8
Vinylite	6000	24.0	2.0
Vinyl chloride	6150	19.4	2.7
Glyptal	6050	18.1	4.0
Gelatine	6100	16.4	4.8

TABLE III. *Relation between film thickness and fluorescence yield*

Film Thickness cm	Peak λ	Fluor. as % of Radiant Energy	Quantum Yield
$0.3(10)^{-3}$	5880A	5.3	4.3
.4	5880	10.6	3.5
.5	5900	14.7	2.4
.6	5900	10.7	3.4
.7	5930	18.6	2.0
.8	5870	16.3	2.3
.9	5900	12.6	3.0
1.0	5900	12.9	3.0
1.5	5920	16.3	2.5
2.3	5970	24.6	1.8
3.0	6000	26.7	1.8
7.3	6000	36.6	1.7
7.9	6030	35.7	1.8
14.0	6050	44.0	1.6
33.0	6150	34.8	2.2
35.5	6200	35.4	2.0
76.1	6250	26.6	2.8

[9] Schmidt, Ann. d. Physik **65**, 247 (1921).
[10] Kautsky, Hirsch and Baumeister, Ber. **64B**, 2053 (1931).

upon the added rhodamine content which accompanies the increase in thickness. Beyond this point the factor of internal absorption of fluorescence within the film becomes continually more noticeable, so that a corresponding decrease occurs in the amount of fluorescence actually emitted. A further evidence of this absorption is the displacement of the fluorescence peak to longer wave-lengths with increase in thickness, a phenomenon similar to that observed with increase in concentration. The accompanying decrease in efficiency however is only slight, for the spacing between molecules remains unaltered and consequently the resonance effect of collision does not become serious.

Variation in light intensities ranging from 300 to 3000 ft. candles failed to have any effect upon the fluorescence yield.

Several other organic fluorescent dyes were investigated as solid solution in cellulose acetate films. Regardless of color, rhodamine proved to be much more efficient. The two which approached it most nearly were safranine T, with a peak at 5900A and a quantum yield of 6.2, and acridine orange, with a peak at 5850A and a quantum yield of 5.3. Several different commercial derivatives of rhodamine B (Schultz No. 573) were examined such as G (No. 572), 3B (No. 574), and 6G (No. 571). The quantum yield from all was of the same order, even in the case of the various forms of No. 571 which are yellow in color and whose peak fluorescence lies at 5800A.

The excitation of rhodamine B was carried out also under illumination from a sodium lamp. The fluorescence showed the same spectral distribution. The yield obtainable was equally high, though its maximum value was reached only at a greater film thickness.

Fading of Rhodamine Fluorescence

The fluorescence remains constant for films kept in the dark, but under exposure to light of the visible range, all types of the salt eventually showed a decrease in fluorescence. The fading of fluorescence of organic dyes has been studied by others for liquid solutions. A good critique is given by Pringsheim.[11] He attempts to show how the fading can be reduced by decreasing the light

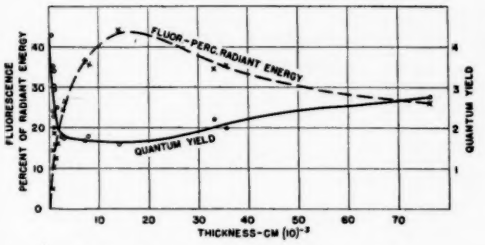


Fig. 2. Relation between film thickness and fluorescence yield. Concentration rhodamine—$4(10)^{-4}$ molar.

intensity or the dye concentration, by lowering the temperature or by changes in the solvent. He ascribes it to the simultaneous activation of two neighboring molecules sufficiently close to one another to react.

In the case of films, the present experience showed that the solvent plays a highly important role. In vinylite and glyptal the fading was extremely rapid; in nitro-cellulose, vinyl chloride and gelatine it was fairly so; and it was least in cellulose acetate and formvar. With the latter two, it was equally as pronounced in vacuum as in air; in fact it was somewhat more rapid in vacuum for the case of cellulose acetate films. As to the color under visible light, dry powder was bleached more rapidly in air than in vacuum; its fading was somewhat more rapid than when in cellulose acetate films.

Quantitative tests on the rate of fading were carried out in cellulose acetate films, by exposing them through clear glass to the visible light from a high pressure mercury capillary lamp at an intensity of 4000 ft. candles, and measuring at intervals the spectral distribution and energy content of the fluorescence. On the assumption that fading involves a progressive destruction of the salt, molecule by molecule, it was calculated that $3(10)^6$ light quanta were required for the destruction of one molecule of rhodamine B. This assumption was supported by the examination of a cross-section of heavy film. The bleaching appeared first in the outermost layer and was separated by a slightly diffuse boundary from the red portion lying further within. As the time of exposure became greater, the boundary progressed continually inward.

This assumption is borne out also by the similarity between fading and the reverse action,

[11] Pringsheim, Zeits. f. Physik **10**, 176 (1922).

namely the increment in fluorescence observed on increasing the amount of rhodamine in a film. The latter change has already been described in connection with the results of Table III. There it was shown that as the film thickness, and therefore the amount of rhodamine per unit of area, was increased, the fluorescence, F, likewise increased as a logarithmic function of the amount of rhodamine, represented by the thickness, x, according to the expression $dF/dx = k(F_{max} - F_x)$. This relation was maintained up to a limiting thickness of $14(10)^{-3}$ cm, a point at which complications arose. The fading of a film lying within this same range of thickness has now been found to follow a similar law, namely, it occurs as a logarithmic function of the time, according to the equation $dF/dt = -kF$. In other words, fading is comparable to a continual thinning of the film and therefore implies a step-wise destruction of molecules.

From this equation, the speed of fading can be expressed as $k = 1/t \log (F_o/F_t)$, where F_o and F_t refer to the fluorescent energies measured initially and at the time t, in hours. In the case of the heavier films of this range, the average concentration of light within the film is reduced by reason of the greater absorption in the outermost layers. Consequently, their rate of fading is less. A typical instance of this effect of thickness is brought out by the results of Table IV, covering samples tested simultaneously to assure uniformity of conditions.

The rate of fading for any one film was fairly constant, as is brought out in Curve A of Fig. 3 for the film $6.3(10)^{-3}$ cm thick. The log of fluorescence is plotted against the time and yields a straight line.

When the critical thickness of $14(10)^{-3}$ cm is exceeded, the rate of fading, expressed similarly, acquired a more complicated form, as represented

TABLE IV. *Relation between fading and film thickness.*

Film Thickness (cm)	Speed of Fading K
$2.1(10)^{-3}$	63
3.1	30
3.8	53
3.8	51
6.3	29
9.4	31
13.4	33

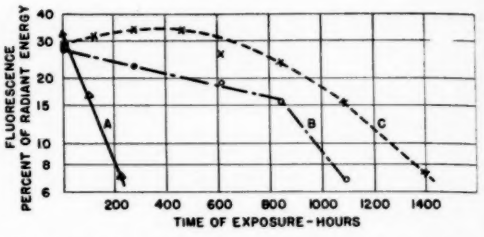


FIG. 3. Relation between rate of fading and film thickness. Rhodamine concentration—$4(10)^{-4}$ molar. A—$6.3(10)^{-3}$ cm thick; B—$61(10)^{-3}$ cm thick; C—$30(10)^{-3}$ cm thick, composite.

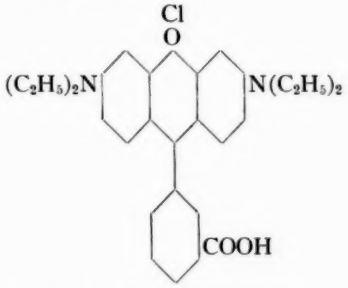
by Curve B of Fig. 3 for a film $61.0(10)^{-3}$ cm thick. In this case, which is typical of most regular films tested at thicknesses of $14(10)^{-3}$ cm and greater, the fading is found to proceed initially at a much slower rate, followed eventually by a rapid rate of the same order as observed for thin films. This is only an apparent discrepancy. The rate of destruction of rhodamine molecules is presumably a constant for films of all thickness. At such heavy thicknesses, however, it is not accompanied by a comparable decrease in the rate of fading. Cross section of films made at various periods during exposure to light have brought out the fact that fading first becomes noticeable as a bleaching of the surface layer, and that this layer is separated from unaltered, red rhodamine below by a diffuse but rather sharp boundary. As the exposure to light continues, this boundary is observed to progress more deeply into the film. Such an action is comparable to a thinning of the rhodamine film. The normal rate of fading which is the measure of the rate of destruction can therefore not appear until the effective thickness has been reduced to $14(10)^{-3}$ cm characteristic of the break in the curve of Fig. 2. The slight decrease in fluorescence observed up to this point is due to absorption effects taking place in the outer layers that have been bleached to a yellow color. Grinding off these layers produced a rise in fluorescence to its normal value.

There is another characteristic of fading which is in accord with this view. It is shown in the curves of Fig. 4 as the changes in spectral distribution which are brought about by exposure to light. Not only does the intensity become progressively less, as denoted by the change in area

under the curve, but the distribution also is changed with a displacement of the peak toward shorter wave-lengths. This is the same trend shown in Table I as the effect of decreasing concentration and in Table III for decreasing film thickness. Its cause is the same, that is, the decrease in internal absorption below 5900A that accompanies the decrease in the amount of rhodamine present, and leads to a greater proportional emission of short wave-length fluorescence.

Very heavy films can be made in which the fluorescence actually increases during early periods of exposure to light. This of course fulfills completely the expectation based on the course of the curve in Fig. 2. It is observed with composite films made by alternate applications of rhodamine dissolved in cellulose acetate and of clear cellulose acetate. The life test of such a film is illustrated by Curve *C* of Fig. 3. The alternate layers in films prepared in this way are not free from rhodamine, for some diffusion necessarily occurs on application of the liquid coat. There is however a periodic variation in the concentration of rhodamine, with as many concentration gradients as there are pairs of coats.

Special derivatives of rhodamine show a considerably greater resistance to fading. The common form of the dye is rhodamine B, Schultz No. 573. It is the salt generally used in publications on the fluorescence of rhodamine. It has the constitutional formula



Substitution for the ethyl groups is apparently without effect. A great improvement however is shown by derivatives in which substitutions have been made for the hydrogen attached to the carboxyl group. Two of these have been tested. One is a rare commercial dye, rhodamine 3B, Schultz No. 574, in which an ethyl group is substituted. The other is a special salt made in this laboratory in which potassium is substituted. It was prepared by adding the equivalent amount of potassium hydroxide followed by several fractional crystallizations in alcohol in which this salt has a much lower solubility than does rhodamine B. Both of these derivatives, especially the potassium salt, showed a very great improvement in life, so much so that the rate of fading was reduced to one-half and less of that found for rhodamine B. Thin films of the potassium salt exposed to the capillary lamp at a distance of 6.3 cm (4000 ft. candles) had lost only 10 percent of their fluorescence after 1500 hours.

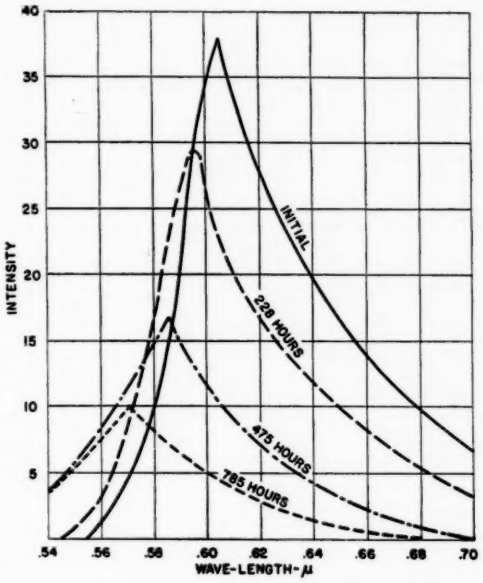


FIG. 4. Decrease in fluorescence during exposure to mercury radiation in visible range. Rhodamine concentration—$4(10)^{-4}$ molar.

From tests on liquid solutions of dyes, Pringsheim[12] concluded that no fading of fluorescence occurred until a critical light intensity had been reached. There was no evidence of this however in the case of solid films. When kept in the dark the fluorescence remained unaltered. Under exposure to light of intensity ranging from 100 to 1500 ft. candles, the rate of fading (K) increased uniformly, not however in direct proportion to the light intensity (I) but rather to its logarithm. The results showed the relation $\Delta K = 5.7\ \Delta \log I$.

[12] Pringsheim, Zeits. f. Physik **10**, 176 (1922).

Summary

The fluorescence of rhodamine was studied in solid films with the aim of finding what factors affect the efficiency of light conversion and its resistance to fading. With regard to efficiency, the most favorable solvent is cellulose acetate, and the most favorable concentration results from the use of a 30 percent solution of this plastic in acetone containing 0.2 gram of rhodamine per liter. Resonance losses account for the abrupt decrease at higher concentrations. The efficiency increases up to a limiting thickness as a logarithmic function of the amount of rhodamine present, and decreases only slightly at greater thicknesses. The fading of fluorescence under exposure to visible mercury light is due to a destruction of rhodamine by bleaching. Its rapidity depends upon the thickness just as does the efficiency—in fact, both follow the same law. Consequently, the longest life is shown by heavy films, having a thickness of $14(10)^{-3}$ cm or greater, and preferably composite, built up of alternate applications of rhodamine and pure binder. The speed of fading depends also on the binder and on the character of the rhodamine. Cellulose acetate and formvar are again the best binders. The vulnerable part of the rhodamine molecule is the carboxyl group, and its resistance to fading can be greatly increased by substituting an organic radicle or a metal for the hydrogen normally attached to it.

The Correction of Aniseikonia with Ophthalmic Lenses

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I. INTRODUCTION

ANISEIKONIA[1] is defined as an anomaly of the binocular visual apparatus in which a difference or incongruity exists between the size or shape of the ocular images of the two eyes.[2] The term "ocular image" describes the final impression received in the higher brain centers through the vision of one eye. The effective magnitude and shape of this impression are determined not only by the properties of the dioptric image as formed on the retina, but also by the modifications imposed upon that image by the anatomical properties and physiological processes by which the optical image upon the retina becomes evident in the higher cortical centers. Aniseikonia may or may not be accompanied by ametropia, that is, refractive errors in the two eyes. If so accompanied the complexity of the problem of correcting both the ametropia and aniseikonia by spectacle lenses is increased. Research has shown, however, that the threshold of discernible size differences between the ocular images is of the order of 0.2 percent; that differences of 1 to 2 percent are sufficient to cause visual, physiological and neurological disturbances; and that differences of the order of 4 percent to 5 percent possibly cannot be tolerated if fusion is to be maintained. The order of magnitude of these differences which formerly would have been ignored, makes it clear that the relative size of the ocular images should be included in the general problem of refraction.

The following paper will describe fundamental concepts regarding the correction of both ametropia and aniseikonia with lenses. The general problem arises when a patient has been corrected in the clinic by trial case lenses and iseikonic test lenses,[3] and it is desired to give that patient a pair of spectacles which can be easily worn and which will be optically equivalent to the prescription found in the testing instrument. The equivalent prescription lenses which must correct both the ametropia and the aniseikonia, because of their particular design, have special magnification properties in addition to their ability to change the vergence of incident light. Their magnification properties are obtained partly through the shape and thickness of the lenses themselves, and partly through the manner of their use before the patient's eyes. The problem will be treated by paraxial ray theory only, for the oblique aberration properties of the lenses may be regarded, it is believed, as being of secondary importance in the correction

[1] Cf. Bibliography.

[2] Under the shape category, aniseikonia also includes declination errors. Such errors can usually be interpreted in terms of a meridional magnification of the dioptric image at some oblique axis.

[3] Zero verging power magnification lenses.

of aniseikonia. The necessity of correcting aniseikonia with ametropia has greatly increased the number of problems in ophthalmic lens design.

Some understanding of the clinical treatment of ametropia and aniseikonia and the problems involved will be presupposed on the part of the reader.

The analysis presented in this paper, developed with the general lens research of the Department, embodies many suggestions of individual members, especially those of Professor A. Ames, Jr., Dr. A. F. Dittmer and Professor G. H. Gliddon. The solution of the problem of correcting aniseikonia with ophthalmic lenses would have been impossible except for the cooperation of the American Optical Company. By instituting new manufacturing methods and technique they have made available accurate lenses with which to carry on and check experimental work. Invaluable suggestions from their scientific staff, especially those from Dr. E. D. Tillyer provided the basis for the approach to the problem as presented in this paper. The practical refinements and improvements of this theory have been made by Dr. Paul Boeder, also of their staff, the result being that the American Optical Company can now efficiently handle the whole field pertaining to iseikonic lenses.

II. The Size of the Retinal Image[4] for a Corrected Ametropic Eye when Single Correcting Lenses Are Used

(a) The formulation for sharp imagery

In the following study of aniseikonia, the first step is to obtain some quantitative relations between the linear size of the object and the linear size of the image upon the retina of a corrected ametropic eye. For the present it will be assumed that the eye is fully corrected, i.e., the dioptric image is sharply formed on the retina. This analysis must then include the properties of the correcting lens. Fig. 1, represents an ametropic eye (in this case hypermetropic) which is directed toward an object O, that lies a distance p (meters) from the cornea of the eye. The ametropia is corrected by a lens L, whose

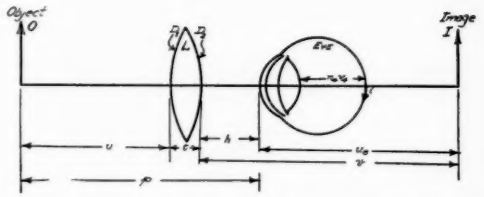

FIG. 1. Diagram used in study of retinal image size when correcting an ametropic eye with a single ophthalmic lens.

ocular surface is a distance h from the pole of the cornea, and which causes an image I of O to be formed a distance u_e (meters) from the cornea. The position of I on the visual axis will be that of the point which is conjugate to the retina and i the retinal image of the object will be sharply formed upon the retina.

The Gauss paraxial ray theory will be used in preference to any other paraxial theory because it readily yields the vertex refraction notation of modern ophthalmic lens practice. Briefly, that theory states that for any symmetrical optical system, conjugate distances[5] for an object and its image measured from the first and last surfaces of the system respectively, are related by

$$Auv+Bv-Cu-D=0, \qquad (1)$$

where u is the reduced object distance measured from the first surface of the system;

v is the reduced image distance measured from the last surface of the system; and

A, B, C and D are the so-called Gauss coefficients which are functions of the various surface powers[6]

D_1, D_2, $D_3 \cdots D_n$, and the reduced separations,

c_1, c_2, $c_3 \cdots c_{n-1}$, between the various surfaces.

The Gauss coefficients can easily be evaluated from the following matrix formulation:[7]

[4] The term "retinal image" as used here, describes that section of the bundle of image forming rays in the vitreous humor of the eye which falls upon the retina, and to which the retinal elements respond.

[5] The sign system used is as follows: light is incident from the left; distances are measured from surfaces; distances to left are negative; distances to right are positive. All separations for purposes of development are taken negative. A reduced distance is an actual distance in an optical medium divided by the index of refraction of that medium.

[6] Surface powers are defined by $D=(n-1)/r$, where r is the radius of the curve in meters and n is the index of refraction.

[7] Cf. T. Smith, "The Primordial Coefficients of Asymmetrical Lenses," Trans. Opt. Soc. **29**, 170 (1928). M. Herzberger, *Strahlenoptik*, Die Grundlehren der mathe-

$$\begin{vmatrix} B & D \\ A & C \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ D_1 & 1 \end{vmatrix} \begin{vmatrix} 1 & c_1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ D_2 & 1 \end{vmatrix} \cdots \begin{vmatrix} 1 & 0 \\ D_n & 1 \end{vmatrix}.$$

The first, third, fifth, etc., matrices represent dioptric surfaces while the second, fourth, etc., represent separations of dioptric surfaces. The number of matrices will depend upon the number of surfaces of the system. The magnification of the system will be given by

$$m = 1/(Au+B) \tag{2}$$

where m is the magnification, i.e., the ratio of the linear sizes of the image and of the object.

Let $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}_4$ be the dioptric powers of the four refractive surfaces of an ametropic eye, and let $\mathbf{c}_1$, $\mathbf{w}$, and $\mathbf{c}_2$ be the reduced center thicknesses of the cornea, anterior chamber and crystalline lens respectively. The reduced image distance v_e from the posterior surface of the crystalline lens for paraxial rays will be given by

$$v_e = (\mathbf{C}u_e + \mathbf{D})/(\mathbf{A}u_e + \mathbf{B}) \tag{3}$$

where u_e is the distance of the object from the first surface, or anterior pole of the cornea, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are the Gauss coefficients of the system, being functions of $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$ and $\mathbf{D}_4$, and $\mathbf{c}_1$, $\mathbf{w}$, and $\mathbf{c}_2$ only. If the optical image lies on the retina, i.e., is sharp, v_e will be the reduced distance between the back surface of the crystalline lens and the retina, and u_e will be the distance from the cornea to the point conjugate to the retina. (cf. Fig. 1.)

The relation between the size of the retinal image i and the effective object size for the eye (here I) will be

$$m_e = i/I = 1/(\mathbf{A}u_e + \mathbf{B}), \tag{4}$$

where m_e is the magnification of the dioptric system of the eye.

Let D_1 and D_2 be the surface powers of the front and ocular surfaces and c be the reduced thickness ($cn = t$) of a correcting ophthalmic lens L. For an object distance u from the front surface of the lens, the image distance v from the ocular surface will be given by

$$v = (Cu+D)/(Au+B), \tag{5}$$

where the Gauss coefficients can be evaluated from

$$\begin{vmatrix} B & D \\ A & C \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ D_1 & 1 \end{vmatrix} \begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ D_2 & 1 \end{vmatrix}, \tag{6}$$

viz.,

$$\begin{aligned} A &= D_1 + D_2 + D_1 D_2 c, \\ B &= 1 + D_2 c, \\ C &= 1 + D_1 c, \\ D &= c, \end{aligned} \tag{7}$$

and $u = p + h + t$.

The paraxial magnification of the lens (m_L), i.e., the ratio of linear image size to linear object size, is

$$m_L = I/0 = 1/(Au+B). \tag{8}$$

Substituting the explicit values for the Gauss coefficients A, B, C, and D into equation (3) and replacing (D_1+U) by F_1, where $U = 1/u$ one obtains for the distance of the image from the ocular surface of the lens

$$v = \frac{1+F_1 c}{F_1 + D_2(1+F_1 c)}. \tag{9}$$

F_1 is the vergence with which the incident rays leave the front surface of the lens. The *vergence power* (V) of the lens for the object distance u, will be defined as the reciprocal of v, *viz.*,

$$V^8 = 1/v = F_1/(1+F_1 c) + D_2. \tag{10}$$

If one puts

$$S = 1/(1+F_1 c), \tag{11}$$

equation (10) becomes

$$V = SF_1 + D_2. \tag{12}$$

Likewise when the explicit values for A and B are substituted into equation (8) and regrouped, one obtains for the magnification of the lens

$$m_L = SU/V. \tag{13}$$

matischen Wissenschaften in Einzeldarstellungen, Bd. **35**, 61 (1931). They may be determined by other methods, e.g., cf. C. Pendlebury, *Lenses and Systems of Lenses* (London, 1884). J. G. Leatham, *Symmetrical Optical Instrument* (Cambridge, 1908).

A has the dimension of a reciprocal of length (since a surface power is the reciprocal of a distance); B and C have no dimensions, being numbers; and D has the dimension of length. These coefficients are related by $BC - AD = 1$.

[8] If such a lens corrects an ametropic eye, for a given object distance, V is the reciprocal of the distance measured from the ocular surface of the lens to the plane that is conjugate to the retina of the ametropic eye.

The ratio of the linear size of the retinal image to the linear size of the object for the correcting lens and the eye in combination can now be found by multiplying equations (4) and (13), i.e.,

$$\frac{i}{0}=m_L m_e=\left[S\frac{U}{V}\right]\left[\frac{1}{\mathbf{A}u_e+\mathbf{B}}\right]. \tag{14}$$

In sharp imagery these two factors are interdependent because the image formed by the lens acts as an object for the eye. This dependence must be made explicit (cf. Fig. 1), using

$$u_e=v+h=(1+Vh)/V$$

and then substituting in (14), whence

$$\frac{i}{0}=\left[\frac{SU}{1+Vh}\right]\left[\frac{u_e}{\mathbf{A}u_e+\mathbf{B}}\right].^{9} \tag{15}$$

In order that the last factor, here, be a function of the eye alone, one eliminates u_e by means of (3) and then has

$$i/0=(SUP)(\mathbf{D}-\mathbf{B}v_e), \tag{16}$$

where $1/P=1+Vh$ and v_e is the reduced distance from the posterior surface of the crystalline lens to the retina, i.e., the reduced depth of the vitreous humor. If the right-hand member of equation (16) is multiplied and divided through by p, the distance from cornea to visual object, then the equation becomes

$$\frac{i}{0}=\left[SUpP\right]\left[\frac{\mathbf{D}-\mathbf{B}v_e}{p}\right]. \tag{17}$$

Now the factor

$$M=SUpP \tag{18}$$

may be written as the product of three component magnification factors, *viz.*,

$$M=S_oLP,^{10} \tag{19}$$

where $S_o=1/(1-D_1c)$, which will be called the shape factor,[11]

$L=p/(u+S_oc)$, which will be called the distance factor,

$P=1/(1-Vh)$, which will be called the power factor,

$V=V_o-S_o^2/(u+S_oc)$, which will be defined as the *vergence power* of the lens for the object distance u $(u=p-h-t)$ (this is identical with vertex power when p becomes very large)

and $V_o=S_oD_1+D_2$ is the *vertex* (back focal) *power* of the lens.

This magnification factor is then equivalent to that for the paraxial angular magnification of any single lens when referred to a specified point on the axis. The point in this case is the pole of the cornea. Thus the ratio of the sizes of the retinal image and of the object for a fully corrected eye may be written

$$i/0=ME, \tag{20}$$

where M is a function of the corrective lens $=(S_oLP)$, E is a function of the dioptric constants of the eye $=(\mathbf{D}-\mathbf{B}v_e)/p$.

Equation (20) and subsidiary relations are the fundamental paraxial ray statements for the angular magnification and power of all single ophthalmic lenses. Their importance will become more apparent in the following discussions.

(b) Other formulations

The particular formulation given above for the size of the retinal image to that of the object for the corrected ametropic eye, as previously stated, is not the only one. The object and image relations as well as the magnification relation can be referred to any pair of conjugate points of the system, whose magnification ratio m is known or determinable. These relations are

$$s'=m^2s/(1+mAs) \tag{21}$$

and

$$m_e=m/(1+mAs), \tag{22}$$

where s and s' are the object and image distances, respectively, measured from conjugate points whose magnification ratio is m.[12] A is the power of that system (the first Gauss coefficient). When used in the above derivation the dependent

[9] Though this division of the magnification into factors for the eye and correcting lens is the most straightforward from the point of view of geometrical optics, it is not the only one, as will be shown later.

[10] Here t, c, h and u are taken positive.

[11] This factor is designated the "effectivity factor," by T. Smith. Cf. Trans. Opt. Soc. **26**, 31 (1924).

[12] These relations, easily found by applying the Gauss theory to the object and image distances referred to a second pair of conjugate points, are identical with the vergence equations of A. Gullstrand. Cf. Helmholtz, *Physiological Optics*, Eng. Trans., Vol. I, p. 277ff.

relationship between the ophthalmic lens and the eye is

$$s=(1/V)+h_x,$$

where $-h_x$ is the distance from the ocular surface of the ophthalmic lens and the first or object point of the reference conjugate points. Then the relation between retinal image size and object size would follow (cf. Eqs. (13) and (16)):

$$i/0=(S_oL_xP_x)(s'/mp_x), \tag{23}$$

where L_x and P_x are the distance and power magnification factors of the lens magnification referred to the first reference point, whence again

$$i/0=M_xE_x. \tag{24}$$

Only the conjugate points for three particular cases need be mentioned *viz.*,

(a) The principal points of the eye, when $m=1$.
(b) The nodal points of the eye, when $m=n/n'$, n and n' being the indices of refraction of air and the vitreous humor, respectively.
(c) The entrance and exit pupils, or stop points.

From a theoretical point of view, in sharp imagery, no single pair of reference points offers advantages. Calculation through lenses and a schematic eye entails nearly the same amount of work.

The selection of the principal points has the advantage that object distances are more accurately expressed as vergences relative to the eyes. The selection of nodal points offers an advantage in geometrical construction. Both sets necessitate linear measurements to fictitious mathematical points. Entrance and exit pupils[13] however, have a special property which makes their selection important.

(c) Formulation for blurred imagery

In practice, it is difficult to determine whether or not the images on the retinas are actually sharp. Moreover to what extent an image can be blurred and good acuity still be obtained or what effect the blurring itself has upon the nature of the retinal image is somewhat indeterminate. Because of these facts it is imperative that some information of the nature of aniseikonia with blurred imagery be obtained.

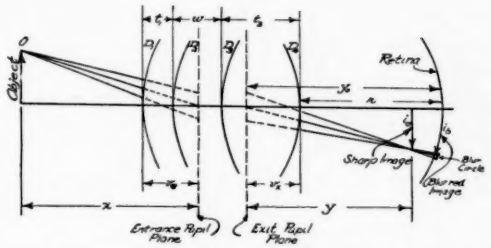

FIG. 2. Diagram used in study of retinal image size with blurred optical imagery.

In Fig. 2 let O be a luminous point in space, near the axis of the eye, which forms an image i in the vitreous chamber. The light energy which is active in the image formation is confined to a cone limited by the entrance pupil of the eye. The emerging cone of light in the vitreous chamber is likewise limited by the exit pupil. Paraxial ray theory requires that the pupillary size be assumed small.

The center of the cone of emergent light in the image space is the ray determined by the center of the exit pupil and the image itself. If the aberrations of the system are not unsymmetric, this center ray marks that part of the cone where maximum light intensity may be expected. Because the entrance and exit pupils lie in conjugate planes, it follows that the object ray which is conjugate to the image ray of maximum light intensity is the ray determined by the luminous point and the center of the entrance pupil.[14] If the image of the luminous point does not lie upon the retina of the eye, a luminous disk or blur circle falls on the retina. The size of this blur circle is determined by the size of the exit pupil and the distance of the sharp image from the retina. Assuming no irregular aberrations are active, the center of this blur circle will define the point of maximum light intensity,[15] and hence

[13] The entrance pupil of the eye is the image of the real pupil formed by that part of the dioptric system which lies between the real pupil and the object. The distance of the entrance pupil from the pole of the cornea can be measured by means of a corneal microscope. The exit pupil will be the image of the real pupil formed by that part of the dioptric system lying between the true pupil and the retina.

[14] This is essentially correct as concerns the human eye—see unpublished work, *Determination of the Effective Entrance Pupil*, by K. N. Ogle, October, 1933.

[15] This may not be strictly true in cases of large spherical aberration. It must be borne in mind that this discussion pertains to centered systems. Actually the pupil is decentered and the dioptric system of the eye is both decentered and tipped relative to visual axis.

the point of effectivity. The distance from the axis of the system to the center of the blur circle will be taken as the size of the blurred image. The nature of aniseikonia with blurred imagery readily follows.

Since the entrance and exit pupils (or stop points) are conjugate points of the ocular system, the image Eqs. (21), (22) and (23) are applicable.

If y is the reduced distance from the plane of the exit pupil to the sharp image in the vitreous humor for a partially corrected ametropic eye, the ratio of the size of this image to the object will be (23)

$$i/0=(S_oL_eP_e)(y/mp_e), \tag{25}$$

where the terms have the meanings given above and where the subscript e indicates that the implicit distances involved are measured from the entrance pupil; m is the magnification ratio of the entrance and exit pupils (a function of the dioptric constants of the eye).

The size of the blurred image on the retina relative to the sharp image will be given by

$$i_b/i=y_o/y, \tag{26}$$

where y_o is the reduced distance from the exit pupil to the retina. The size of the blurred image on the retina relative to the same object size above will be given by multiplying Eq. (25) by (y_o/y). One obtains

$$i_b/0=(S_oL_eP_e)(y_o/mp_e) \tag{27}$$

or $$i_b/0=M_LE_b,$$

where M_L is the angular magnification of the lens referred to the entrance pupil of the eye

and $E_b(=y_o/mp_e)$ constitutes the ocular dioptric part of the retinal image magnification analogous to the factor E (page 326) for the sharp imagery detail.

In comparing the relation for sharp imagery (Eq. (19)) and that above, it must be borne in mind that in sharp imagery there is a strict dependent relationship between the dioptric factor $(1/(1-Vh))$ and the dioptric factor $(\mathbf{B}v_e-\mathbf{D})$ in a given eye, whereas in the expression above the two factors may be quite independent. It does not furnish a criterion as to the nature or amount of blur in each eye. If the magnitude of V is such that the ametropia becomes completely

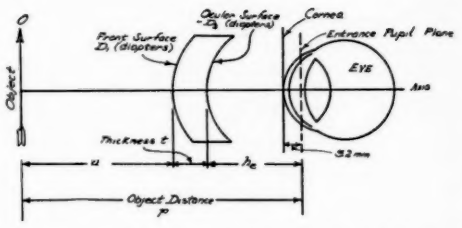

FIG. 3. Schematic representation of an ophthalmic lens used before the eye showing notations used in text.

corrected, then the optical image on the retina becomes sharp and the two expressions must be identical. In this connection it should be pointed out that if the images formed on the retinas are somewhat blurred, but those blurs are a constant condition for each eye, there will be again a strict relationship between the ametropia of each eye and the power of the correcting lenses. The difference between the nature of the blur in each eye will have significance in the magnitude of aniseikonia. This difference will be included in values of y_o and m for each eye.

The pupil of the eye lies in contact with the anterior surface of the crystalline lens which changes position only slightly during accommodation, consequently the magnification of the entrance and exit pupils (m) is only slightly affected. Hence the entrance pupil becomes important as a point of reference.

The magnification, M_e, for the ophthalmic lens as given above is the angular magnification of a lens referred to the entrance pupil of the eye, namely

$$M_e=S_oLP, \tag{28}$$

where $1/S_o=1-D_1c$ a shape factor,

$L=p_e/(u+S_oc)$ a distance factor,

$1/P=1-Vh_e$[16] a power factor,

$V=V_o-S_o^2/(u+S_oc)$, a relation between vergence power and vertex power and object distance,

$V_o=S_oD_1+D_2$, the vertex power,

and $h_e=$ the distance from the ocular surface of the lens to the entrance pupil of the eye.

[16] This relation differs from that of Eq. (19) in that h_e here is measured from the entrance pupil instead of the cornea. In the schematic eye the entrance pupil lies about 3.2 mm behind the cornea. Because this expression is best adapted for either blurred or sharp imagery it is perhaps the best to use empirically.

Cf. Fig. 3. In these relations the signs of t, h and p are reversed so that the first two are positive and p becomes positive for real distances of the object before the eye.

The general relation for the angular paraxial magnification of a single ophthalmic lens, as given by Eq. (28) can, by eliminating the vergence power, be written

$$M = S_o P_o L_o, \tag{29}$$

where $S_o = 1/(1 - D_1 c)$, shape factor,

$P_o = 1/(1 - V_o h)$, power factor,

$$L_o = \frac{p}{p + h(S_o^2 P_o - 1) - t((n - S_o)/n)},$$

distance factor,[17]

$V_o = S_o D_1 + D_2$, the vertex power, diopters.

The object distance from the reference point taken positive, is designated p; h is the distance from the reference point to the ocular surface of the lens; D_1 and D_2 are the front and ocular surface powers, and c is the reduced thickness, being equal to t/n, where t is the axial thickness and n the index of refraction. This particular formulation obviates the necessity of computing the vergence power of the lens and is in general more suitable for calculation purposes. The only unavoidable and undesirable feature is that the distance factor L_o includes both the shape and power factors.

The shape factor, S_o, will vary from zero percent for a lens with a plano front surface to 3.25 percent for a lens with a 16 diopter front surface and 3 mm thickness. The distance factor usually will vary from zero for large visual distances to a few tenths of a percent for reading distances. The power factor will amount to about 1.5 percent per diopter of power of the correcting lens.

III. Aniseikonia

Aniseikonia being defined as an anomaly of the binocular visual function in which a difference exists between the sizes of the ocular images of the two eyes, one should find analytically the relation between the ocular image size and linear retinal image size and finally the relation between the sizes of the ocular images of the two eyes. Since the ocular image is the result of a subjective impression, it is apparent that its absolute size cannot be determined. However, if it were possible to measure the ocular image size one could compare that size with the size of the retinal image and write

$$i' = ki, \tag{30}$$

where i' is the size of the ocular image;

i is the size of the corresponding retinal image; and

k is the constant of proportionality. While k, for one eye, need not be constant for all conditions under which the eyes may be used, it is reasonable to suppose that for any fixed condition of the eyes in binocular vision k does not vary.

Thus for the same sized object, one can write that the ratio[18] of the sizes of the ocular images of the two eyes compared to that of the retinal images of the two eyes is

$$\frac{i_1'}{i_2'} = \frac{k_1 i_1}{k_2 i_2} = \left[a\right]\frac{i_1}{i_2}, \tag{31}$$

where $a = k_1/k_2$ is the ratio of the constants of proportionality between ocular image sizes and retinal image sizes. The ratio $[a]$ is a measure of that difference in size between the ocular images due to physiological and anatomical differences in the visual apparatus of the two eyes. This quantity has reality and can, under certain conditions be measured. In a given pair of eyes where $[a] \neq 1$, even if the retinal images were equal in size, a difference would exist between their apparent sizes. The ratio $[a]$ may be designated a "basic aniseikonia." There is no reason for assuming that $[a]$ is the same for all corresponding parts of the two retinas. For the present discussion it will be necessary, however, to assume its constancy for any given position of the eyes, and for paraxial rays.

[17] If $V_o = 0$, $D_1 = 0$, L_o = angular magnification of plane parallel of thickness t mm.

[18] Comparing the ocular images in any other manner explicitly involves the absolute magnitude of one of the k's. The ratio permits one to compare the sizes in percentages.

From Eq. (20) above, it follows that for a visual object of size O, at a distance p from the cornea, viewed by a fully corrected eye, the size of the retinal image would be

$$i = MEO,$$

where $M = S_o LP$ a magnification expression being a function of the correcting lens only and $E = (\mathbf{D} - \mathbf{B}v_e)/p$ a magnification expression for sharp imagery, being a function of the constants of the ametropic eye only or $E = y_o/m$ for blurred imagery. Thus, for the two eyes viewing the same object[19] in binocular vision the actual retinal image sizes will be

$$\begin{aligned} i_1 &= M_1 E_1 O \\ \text{and} \quad i_2 &= M_2 E_2 O. \end{aligned} \tag{32}$$

The ratio of the actual sizes of the retinal images of the object O, upon the two retinas will be

$$\frac{i_1}{i_2} = \left[\frac{M_1 E_1}{M_2 E_2}\right] = \left[\frac{M_1}{M_2}\right]\left[d\right], \tag{33}$$

where d expresses the ratio of E_1 to E_2. But because of the anatomical and physiological visual differences, i_1 will appear to the individual to be of different size than i_2 according to

$$\frac{i_1'}{i_2'} = \left[\frac{M_1}{M_2}\right]\left[d\right]\left[a\right]. \tag{34}$$

The ratio of the sizes of the ocular images, i_1'/i_2' is designated the *eikonic ratio*, R_o, and can be measured on a testing instrument.

It must be clear that in sharp imagery, the exact magnitudes of M_1 and M_2 and hence their ratio will differ, depending upon the particular pair of points in the eye selected as reference points from which object and image distances are measured. In all cases, however, irrespective of the mathematical formulation, or reference points selected in a given pair of eyes, the ratio of image size to object size will be

$$i/0 = (ME),$$

where M and E refer to component magnification factors. For any given corrected ametropic eye, (ME) will be a constant quantity. If, in the formulation, M, the magnification of the corrective lens, be changed through a change in the reference system, the ocular magnification, E, must change inversely. While the magnitude of the eikonic ratio (R_o) has reality, the ratio of M_1 to M_2 will depend upon the reference points chosen. Only the ratio

$$\left[\frac{M_1 E_1}{M_2 E_2}\right]$$

is real. Therefore the *ratio of the magnifications of the ophthalmic lenses or test lenses alone has no inherent meaning so far as aniseikonia is concerned*. The significance of the ratio appears only when the equivalent iseikonic prescription is determined, (Eq. (46)). It is necessary that the same reference points be used in designating both the magnifications of the test lenses and those of the ophthalmic iseikonic lenses.

This discussion pertains *only* to the case in which sharp images are maintained on the retinas of the two eyes. Then the choice of reference points is arbitrary. However, if by any chance errors occur in the correction of the ametropia, the images will be blurred. Then there is only one pair of reference points which is strictly valid, namely the entrance and exit pupils. This restriction arises from the very nature of image formation and is not dependent on the method of analytical description. In sharp imagery any pair of points may be selected for reference, but for blurred imagery the entrance and exit pupils must be used. Therefore the latter, which will be correct for either sharp or blurred imagery, are the advantageous points to select for all general formulation.

It should be pointed out here that in certain cases, the ratio of the sizes of the ocular images alone will not define all the incongruities between the ocular images. For example, the change in declinations introduced by meridional iseikonic corrections whose axes are oblique, presents additional problems not entirely included in the ratio comparison.

IV. Two Element Correcting Lens System—Trial Case Lenses

In the clinic a patient's ametropia is corrected by means of trial case lenses, which consist of spherical and cylindrical lenses made in con-

[19] As would be the case in the testing instrument.

venient steps of power. The spheres are used to determine the amount of spherical ametropia and the cylinders to determine the amount and axes of astigmatism. Quite generally in a given examination both a spherical and a cylindrical lens will be needed before each eye to fully correct the ametropia present. Therefore the power and magnification relationships for a two element lens system are necessary for an understanding of the correction of aniseikonia.

Let the four successive surface powers of the two lenses be D_1, D_2, D_3 and D_4 diopters respectively, and c_1, w and c_2 the reduced thickness of the first lens, the air separation, and the reduced thickness of the second lens respectively (in meters). (See Fig. 4.) Referring to the preceding material leading up to equations (12) and (18) and to the following figure, it is clear that one need only evaluate the Gauss coefficients A, B, C and D for the four surfaces of a two lens system. The vergence power of that system for a specified object distance will be given by (cf. Eq. (5)),

$$V_t=(Au+B)/(Cu+D), \tag{35}$$

and the magnification of the system will be given by

$$m_L=I/0=1/(Au+B). \tag{36}$$

When the equivalents for the Gauss coefficients[20] are substituted in equation (35), and the terms regrouped one can show that the *total vergence power* (V_t) of the system will be

$$V_t=V_{o2}+\frac{V_1S_{o2}^2}{1+V_1(w+S_{o2}c_2)},^{21} \tag{37}$$

where $V_{o2}=S_{o2}D_3+D_4$, the true *vertex power* of the second lens,

$1/S_{o2}=1+D_3c_2$,

$V_1=S_1(D_1+U)+D_2$, the *vergence power* of the first lens for the object distance u from the front surface of that lens,

$1/S_1=1+(D_1+U)c_1$,

$U=1/u$

and where the subscripts (1) and (2) apply to the first and second lenses respectively.

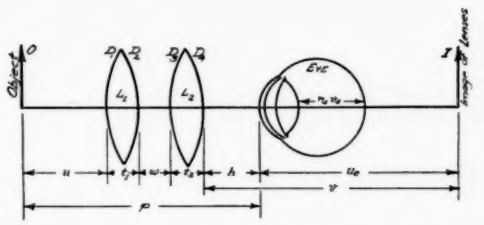

FIG. 4. Diagram used in study of retinal image size when correcting an ametropic eye with two ophthalmic lenses.

It can then be shown that the lens magnification of the system will be

$$m_L=(S_1S_{o2}T)(U/V_t), \tag{38}$$

where S_1, S_{o2}, U and V_t have been defined above and where

$$1/T=1+V_1(w+S_{o2}c_2). \tag{39}$$

If the two element lens system be used to correct the ametropia of an eye, one finds that the ratio of the sizes of the dioptric image on the retina and of the object can be written (cf. equations (4), (8) and (14)),

$$\frac{i}{0}=m_em_L=\left[S_1S_{o2}T\frac{U}{V}\right]\left[\frac{\mathbf{D}-\mathbf{B}v_e}{u_e}\right].$$

In sharp imagery, however,

$$u_e=(1+Vh)/V,$$

one can obtain, again

$$i/0=ME, \tag{40}$$

where

$$M=S_{o1}S_{o2}TLP,^{22} \tag{41}$$

$1/S_{o1}=1-D_1c_1$,

$1/S_{o2}=1-D_3c_2$,

$1/T=1-V_1(w+S_{o2}c_2)$,

$L=p/(u+S_{o1}c_1)$,

$1/P=1-V_th$,

[20] The Gauss coefficients for a four surface system are

$A=A_{12}(1+D_4c_2)+A_{34}(1+D_1c_1)+A_{12}A_{34}w$,

$A_{12}=D_1+D_2+D_1D_2c_1$,
$A_{34}=D_3+D_4+D_3D_4c_2$.

$B=(1+D_2c_1)(1+D_4c_2)+(c_1+w(1+D_2c_1))A_{34}$,
$C=c_2A_{12}+(1+D_1c_1)(1+D_3c_2)+A_{12}w(1+D_3c_2)$,
$D=(1+D_2c_1)c_2+c_1(1+D_3c_2)+(1+D_3c_2)(1+D_2c_1)w$.

[21] Cf. T. Smith, "Back Vertex Powers," Trans. Opt. Soc., 26, 35 (1924), for case when $U=0$, i.e., parallel incident light.

[22] Taking c, h and u as positive here.

where V_t, the total vergence power of the two lenses for the distance u, is given in Eq. (37).

$$V_1 = S_1(D_1 - U) + D_2,$$

$$S_1 = \frac{1}{1-(D_1-U)c_1} = \frac{S_{o1}u}{u+S_{o1}c_1},$$

$$U = 1/u, \quad u = p - h - t_1 - t_2 - w,$$

$$V_1 = V_{o1} - \frac{S_{o1}^2}{u+S_{o1}c_1},$$

$$V_{o1} = S_{o1}D_1 + D_2,$$

$$V_{o2} = S_{o2}D_3 + D_1,$$

and where again $E = (\mathbf{D} - \mathbf{B}v_c)/p$, a function of the dioptric constants of the eye.

V. The Relation between the Magnification Properties of the Test Lenses and Those of the Spectacle Lenses in the Correction of Aniseikonia

(a) Eikonic condition

When the patient is tested in a clinical instrument and the refraction in both eyes has been balanced with certain test lenses, the difference between the sizes of the ocular images in various meridians can be corrected, and hence measured by special iseikonic lenses. This measurement for any given meridian specifies the eikonic ratio R_o described above, *viz.*,

$$R_o = \left[\frac{M_1}{M_2}\right]_e \left[d\right]\left[a\right]. \tag{42}$$

The actual procedure in making the measurements in the clinical instrument, of course, consists in modifying the ratio of the magnifications of the test lenses by means of special size lenses which are added before one or the other of the eyes until the ocular images appear equal. Then

$$\frac{i_1'}{i_2'} = R\left[\frac{M_1}{M_2}\right]_e \left[d\right]\left[a\right] = 1, \tag{43}$$

where R is the ratio of the magnifications of the modifying lenses. Clearly

$$R = 1/R_o.^{23} \tag{44}$$

[23] In general R differs from unity by a few percent, and it has been convenient to express R as $R = 1+e$, and refer to the percent eikonic difference, *viz.*, $100e$.

Equation (43) correctly describes the ocular image size ratio for any type of lenses used with the eyes, consequently it describes the condition when either test or spectacle lenses are used. Inspection shows that M_1 and M_2, which designate the magnification properties of ophthalmic lenses, can be adjusted by careful selection of the shape and thickness of the individual lenses so that the ocular images will be equal, i.e.,

$$\frac{i_1'}{i_2'} = \left[\frac{M_1}{M_2}\right]_s \left[d\right]\left[a\right] = 1. \tag{45}$$

This equation states the necessary condition that corrective ophthalmic spectacles be iseikonic. This condition must be satisfied in all meridians.

It is simple now to state the necessary conditions for determining the equivalent iseikonic prescription. From these the spectacle lenses can be designed and manufactured which correct aniseikonia and ametropia for a given prescription. For any meridian Eqs. (42) and (44) must be identical. Thus one obtains the statement of iseikonic equivalence,

$$\left[\frac{M_1}{M_2}\right]_s = R\left[\frac{M_1}{M_2}\right]_e, \tag{46}$$

where the subscripts e and s refer to the clinical test lenses and spectacle lenses respectively. This relation will be referred to as the iseikonic condition necessary for the design of the equivalent iseikonic prescriptions.

(b) Differential magnification properties

It is not necessary to deal with the total magnifications, but with the differential part of the magnifications between the spectacle lenses and the lenses of the clinical instrument. If one substitutes the component factors for the magnifications of Eqs. (41) and (22) for the single (spectacle) and double (trial case) lenses into Eq. (46), one has for a given meridian,

$$\left[\frac{(S_oLP)_1}{(S_oLP)_2}\right]_s = R\left[\frac{(S_{o1}S_{o2}TLP)_1}{(S_{o1}S_{o2}TLP)_2}\right]_e. \tag{47}$$

In order that the ametropia be corrected by both trial case lenses and spectacle lenses, the vergence power must be made the same for

each eye. Since the P magnification factor is defined by

$$1/P=1-Vh,$$

where V is the vergence power and h is the distance from the entrance pupil to the ocular surface of the correcting lens, the P factors for the spectacle lens and trial case lenses for the same eye will differ only by the difference in h in the two cases. In general the h_s for a spectacle lens will differ from h_c for the test lens by a small distance (Δh), where

$$h_s=h_c+\Delta h. \tag{48}$$

Equation (47) can be divided through by $P_1/P_2]_c$, and one obtains

$$\frac{(S_oLP')_1}{(S_oLP')_2}\Big]_s=R\frac{(S_{o1}S_{o2}TL)_1}{(S_{o1}S_{o2}TL)_2}\Big]_c, \tag{49}$$

where $$P'=\frac{1}{1-V(\Delta h)},$$

when (Δh) is small.

One can further reduce the total magnification by multiplying the L factors on both sides of Eq. (47) by p_o/p where p_o is the object distance measured from the ocular surface position of the trial case lenses, i.e.,

$$p=p_o+h_c. \tag{50}$$

One then defines the differential paraxial angular magnification of a single ophthalmic lens, as

$$M_a=S_oLP', \tag{51}$$

where $$S_o=1/(1-D_1c),$$

$$L=p_o/(u+S_oc),$$
$$P'=1/(1-V(\Delta h)),$$
$$u=p_o-t-\Delta h.$$

In general the greater part of the total angular magnification of an ophthalmic lens of any power is due to the P factor magnification. Hence differential magnification deals with smaller quantities and indicates more directly the magnification characteristics of the ophthalmic lens.

The vergence powers of the ophthalmic lenses[24] and the vergence powers of the trial case lenses must be identical in all meridians for the same eye, thus one has for the statement of ametropic equivalence,[25]

$$(V_1)_s=(V_1)_c/(P_1'),$$
$$(V_2)_s=(V_2)_c/(P_2'). \tag{52}$$

Equations (46) and (52) state the necessary conditions for the design of the lenses for the equivalent iseikonic prescription in any specified meridian.

(c) Cylindrical lens excess magnifications

From what has been written above concerning the eikonic condition, e.g., Eq. (49), it is clear that the magnifications needed in the test instrument to equalize the sizes of the ocular images (expressed by R), are not a simple measure of the aniseikonia present, because of the magnification properties of the test lenses. These magnifications are unavoidable and are variable and of considerable magnitude. They will be dealt with specifically. A redesign of the trial case lenses may simplify these magnifications.

In testing the ametropia with standard trial case lenses set it is common practice to place the spherical refractive lens nearest the eye and the cylindrical refractive lens farthest from the eye. Obviously it is an advantage, to have the power of each cylinder so marked that its effective power is directly additive to the power of any sphere. Then, with any combination of sphere and cylinder the total effective power can be easily found by simple arithmetic. Inspection of Eq. (37) indicates the manner in which this may be done. For distant vision when the test distance becomes great, the total effective vertex power V_{ot} (diopters) becomes

$$V_{ot}=V_{o2}+\frac{V_{o1}S_{os}^2}{1-V_{o1}(w+S_{os}c_2)}, \tag{53}$$

where now V_{o1} is the vertex power of the cylindrical lens. The powers of the cylinders and spheres can be made additive if the marked power, V_1', of the cylindrical lens be defined by

$$V_{o1}'=\frac{V_{o1}S_{os}^2}{1-V_{o1}(w+S_{os}c_2)}. \tag{54}$$

[24] Referred to the points before the eyes at which the vergence powers of the trial case lenses were specified.

[25] These relations follow since the positions of the images formed by the spectacle lenses must be the same as were the positions of the images formed by the corresponding trial case lenses.

Then for distant vision

$$V_{o1} = V_{o2} + V_{o1}'.^{26} \qquad (55)$$

Inspection will show, however, that if any cylindrical correcting lens be added to any spherical correcting lens and the total effective power be accurate, the front curves and thicknesses for all the spherical lenses must be the same.

When trial case lenses manufactured in this manner are used in testing aniseikonia, the shape factor S_{o2} in Eq. (49) is always the same and hence will drop out of the ratio of the magnifications of the trial case lenses (Eq. (49)). The remaining quantity for each cylindrical lens $N = S_{o1}LT$, remains.[27] This quantity, N, designated as the cylindrical excess magnification, depends only upon the particular trial case cylinder used in the testing instrument. It is an unwanted magnification factor which may artificially create a difference in size between the ocular images in the power meridian of each cylindrical trial case lens used. The eikonic ratio as determined in the clinic cannot then indicate the true aniseikonia. Under certain circumstances it is possible[28] to design a trial case system the use of which would eliminate this factor. In such a case the ratio between the ocular images as measured in the clinic would have quantitative significance regarding the amount of aniseikonia and would facilitate the design of the equivalent prescription lenses.

VI. Types of Correction

In general it has not been feasible or practical in the clinic, to measure aniseikonia in all meridians and the measurements have been limited to the vertical and horizontal. These meridians in which aniseikonia has been measured in the clinic are designated eikonic meridians.

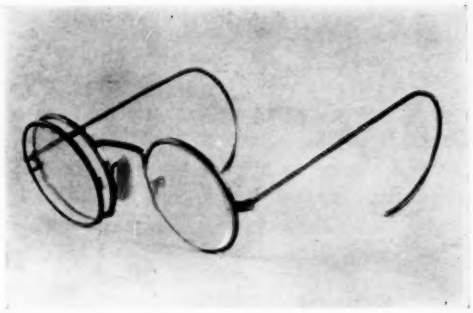

FIG. 5. Illustration of spectacles for the correction of aniseikonia using a fitover lens.

The problem of designing the equivalent spectacle lenses for the lenses in the test instruments, resolves itself into satisfying the eikonic condition in the eikonic meridians and the ametropic condition in the principal power meridians of the two eyes. For distant vision, for example, these two conditions are

$$\frac{(S_oP')_1}{(S_oP')_2}\Big]_s = R\frac{N_1}{N_2}\Big]_e \qquad (56)$$

and

$$\begin{aligned}(V_o)_1]_s &^{29} = V_o]_e/P']_1,\\ (V_o)_2]_s &= V_o]_e/P']_2.\end{aligned} \qquad (57)$$

In general the axes of astigmatism present in any examination may not coincide with the eikonic meridians. In those cases where the axes of astigmatism do coincide with the eikonic meridians the problem of designing the complete equivalent lens is readily solved. Where the axes do not coincide the problem is more difficult, and its solution may even require double toric lenses with non-parallel axes.

(a) Fitover type equivalent prescriptions

From the point of view of simplicity it is easy to make up equivalent prescriptions by using standard ophthalmic lenses to correct the ametropia and using zero verging power fitover lenses to correct overall and meridional eikonic differences. These fitover lenses are mounted in separate eyewires and are held before one or both

[26] This additive property is found in most of the trial case sets manufactured today.

[27] The L factor is unity for distant vision and hence does not enter into the cylindrical excess magnification. If cylindrical lenses are included before both eyes in the test, even though one is of zero power, the L factor drops out of the eikonic trial case lens ratio.

[28] One means for doing this is to design the shape of the cylindrical lens for distant vision so that its shape factor S_{o1} just offsets the T factor, i.e., $S_{o1}T = 1$. A different set of cylinders would be required for near vision unless cylindrical lenses are always placed before both eyes (though one may be of zero power).

[29] In terms of the surface powers and thickness of the lens the vertex power (cf. Eq. (19)) can be written $V_o = D_1 + D_2 + e$, where e is an allowance factor equal to $S_oD_1^2c (= D_1^2c$, approximately) that can be found from prepared tables.

spectacle lenses by spring-clips (Fig. 5). This method is especially suitable for those cases of oblique astigmatism where the astigmatic axes do not coincide with the eikonic meridians. The method is also suitable for the clinic where trial prescriptions may be needed. Tables can be prepared in which the magnifications in the principal meridians of each ophthalmic lens are listed. To avoid an allowance for the cylindrical excess magnifications it is convenient to define the magnification in each principal meridian as entered in the table as (cf. Eq. (56))

$$M' = S_o L P'/N. \tag{58}$$

In cases of oblique astigmatic axes it is necessary to find the magnification components in the eikonic meridians. Then the equivalent prescription in each eikonic meridian must be satisfied by

$$M_1'/M_2']_s F = R, \tag{59}$$

where M_1' and M_2' are the magnifications of the two lenses defined by Eq. (58); F is the fitover zero verging power lens correction and R is the ratio of the iseikonic test lenses used in a given examination to equalize the ocular images. In terms of percent magnification the above relation states that the difference between the difference in size of the ocular images as measured in the clinic and the difference of the magnifications as defined by Eq. (58) of the standard correcting power lenses is corrected by suitable fitover lenses.

(b) Unitary prescriptions

The fitover type spectacles are not always desirable, however, because of their excessive weight, difficulty to keep clean, and for cosmetic reasons; hence lenses are desired which include both ametropic and aniseikonic corrections, i.e., unitary lenses. From the above theory the important means available to meet the eikonic corrections are:

(1) Bending, or altering the shape of the lenses, by varying the front surface power.
(2) Changing the axial thickness.
(3) Varying the distance of the lens from the eye.

It is possible to select the curves, thicknesses and positions for the two corrective lenses to duplicate any measured eikonic ratio, provided that ratio is not too large. Where the eikonic ratio is different in the two eikonic meridians this requires a bi-toric lens, and the means of obtaining different magnifications are limited, since (2) and (3) above must remain the same for both.

Inspection of Eq. (49) shows that some simplification of design can be obtained if the spectacle lenses are mounted before the patient's eyes so that the ocular surfaces are the same distances from the eyes as were the corresponding trial case lenses in the clinical instrument. Then $\Delta h = 0$, and the power factor (P') in the magnification disappears.[30] In general the method is not feasible, because of practical problems in fitting special type frames, or in designing excessively thick lenses.

At present it is practical to specify the position of the spectacles before the patient's eyes by the distance from the pole of the cornea to the center of the eyewire of the frames. From the point of view of the equivalent lens design this complicates the problem since (Δh) is usually unknown beforehand. Its magnitude depends upon the lens shape, thickness, and size and the nature of the bevel. Various technical means are available, however, for solving the problem, but cannot be considered here.

In many cases when the eikonic ratio is large or quite different in the principal meridians, doublet correcting lenses may be necessary. These doublets act as airspaced telescopic units but include the refractive correction. They may be designed to fit into ordinary eyewires. They provide a means for obtaining higher overall and meridional magnifications than can be obtained practically with single lenses.

VII. Zero Verging Power Iseikonic Lenses

It is sometimes convenient to refer to the "verging power" of an ophthalmic lens in describing its power behavior in a given case. By "verging power" will be meant the change in the vergence of incident light caused by the power of the lens. It must refer to a specific object distance. It is defined algebraically as

$$\mathbf{V} = V + 1/(p - h), \tag{60}$$

[30] Dr. E. D. Tillyer first pointed out this simplification.

where **V** is the verging power for a given object distance (diopters),
V is the vergence power for the same given object distance,
$p-h$ is the distance from the object to the ocular surface of the lens.

Were the lens infinitely thin the verging power would be identical with the *vertex power*, as can be seen by substituting the equivalent expression for the vergence power cf. equation (60) above. If the incident light is from an infinitely distant object, the *verging power* is identical with the vertex power. A lens which has zero verging power for a given object distance is one in which the emergent light has the same vergence as the incident light, i.e., the position of the image of an object formed by the lens coincides with that of the object itself. Such a lens would usually have magnifying properties, however.

Zero verging power lenses have played, and are playing, an important role in the measurement and correction of aniseikonia. These lenses have been used for two purposes, namely to measure, in the clinic, the magnitude of aniseikonia and second, to correct the aniseikonia found. Corrections up to four or five percent overall and meridional magnification can be had with these single zero verging power lenses. These lenses may be designed by means of equation (19). Such a lens for a given object distance p and ocular surface distance h, will have a *vergence power* equal to the reciprocal of the distance of the object from the ocular surface of the lens, i.e., $V=-1/(p-h)$, then

$$M=S_o\frac{p-h}{p-h-t+S_oc},$$
$$V_o=\frac{S_o^2}{u+S_oc}-\frac{1}{p-h}, \tag{61}$$

where $V_o=S_oD_1+D_2$.

These two equations contain three unknowns, D_1, D_2 and t, hence one must be specified.

Arbitrarily specifying the thickness for any desired lens one can find the front and ocular surface powers, by solving the two equations for D_1 and D_2. One has

$$D_1=\frac{n((M-1)/M)(p-h)-(n-1)t}{t(p-h-t)}$$
$$\text{and}\quad D_2=-\frac{n(p-h-t)(M-1)-t(n-1)}{t(p-h)}. \tag{62}$$

If such a lens is designed to have zero verging power for an infinitely distant object, the design follows

$$M=S_o=1/(1-D_1t/n), \tag{63}$$

whence $D_1=(M-1)n/Mt$

or $M=1-D_2t/n,$

$$D_2=-MD_1=-(M-1)n/t.$$

Such a lens is independent of h, the distance from the cornea. The lenses for the most practical set are designed for a visual distance of 75 cm.

It is clear that a lens can have zero verging power for one visual distance only and when used for any other visual distance the lens will have a small amount of verging power. If it be assumed that the eyes do not change accommodation to make up for this small amount of power it is possible to design a lens which has zero verging power at one visual distance, but will have the same angular magnification at all visual distances. The equation stating this condition can be obtained by differentiating Eq. (19) with respect to p and equating the result to zero. The final relation, *viz.*,

$$h(MS_o-1)-(t/n)(n-S_o)=0 \tag{64}$$

is independent of p. This relation together with Eq. (61) uniquely determines D_1, D_2 and t for given values of p, h and M. One finds for the thickness

$$t=\frac{nh(M^2-1)(p-h)}{h(M^2-1)+p(n-M)}. \tag{65}$$

D_1 and D_2 are found from Eqs. (62).

Overall and meridional zero verging power magnification lenses are needed. In order that the meridional lenses have unit magnification for a given visual distance in the no magnification meridians double saddle-shaped bi-toric lenses are required. The lenses may be ground with plano surfaces in these meridians, and the result-

TABLE I. *Zero verging power lenses.*
Lenses which change the apparent size of the object without materially changing the vergence of the light which enters the eye. Lenses designed to have zero verging power for a visual distance of 75 cm.

Lens No.	Magnification 75 cm	D_1 Front Surface Diopters	D_2 Rear Surface Diopters	t Thickness mm	Verging Power at 6 meters in Diopters
1	0.25%	+2.43	−2.43	1.20	+0.01
2	0.50	+5.51	−5.52	1.22	+0.01
3	0.75	+9.84	−9.89	1.07	+0.02
4	1.00	+10.17	−10.24	1.39	+0.02
5	1.25	+10.47	−10.56	1.68	+0.03
6	1.50	+10.71	−10.82	1.97	+0.04
7	1.75	+10.89	−11.02	2.26	+0.04
8	2.00	+11.03	−11.18	2.55	+0.05
9	2.25	+11.15	−11.32	2.83	+0.05
10	2.50	+11.29	−11.49	3.10	+0.06
11	2.75	+11.38	−11.60	3.38	+0.07
12	3.00	+11.49	−11.73	3.65	+0.07
14	3.50	+11.68	−11.97	4.18	+0.08
16	4.00	+11.82	−12.16	4.70	+0.10
18	4.50	+12.03	−12.43	5.18	+0.11
20	5.00	+12.20	−12.64	5.66	+0.12

Distance of ocular surface of the lens taken as 25 mm from cornea. $n = 1.523$.

ant magnifications for the various visual distances can be measured or computed. Thus the magnification error can be corrected in the prescription. It is sometimes convenient to use compensating lenses placed before the other eye in the test instrument which then balances the magnification effect in the plano meridians of the meridional lenses.

In Table I are shown the specifications for a typical set of zero verging power lenses.[31]

VIII. LOGARITHMIC NOTATION

In practice it is convenient to specify the magnification of an ophthalmic lens in terms of percent magnification, and refer to the amount of aniseikonia as a percent difference in size between the ocular images. This notation in the design of the equivalent prescription may introduce errors, however, since magnification ratios or products would be considered as percent differences or sums. Hence the device is used of expressing the magnification as 100 times the natural logarithm of the magnification, written LM. Then differences and sums are not only convenient and accurate, but are near enough to percentages to have quantitative meaning.[32] Thus the LM value of an ophthalmic lens is Eq. (19)

$$LM = LS_o + LL + LP,$$

where LS_o, LP and LL are 100 times the natural logarithm of the S_o, P and L factors.

The logarithm notation is also convenient in expressing the iseikonic condition (cf. Eq. (46)) for then

$$LM_1]_s - LM_2]_s = LR + LM_1]_c - LM_2]_c.$$

It must be remembered that the above theory is presented from a strict theoretical point of view, and in many instances approximations can be made because of certain tolerances allowable in the measurement of aniseikonia and in the grinding of the finished spectacle lenses.

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[31] These lenses have excellent field properties with negligible distortion.

[32] This follows from the series expansion of $\log_e M$, i.e.,

$$\log_e M = (M-1) - \tfrac{1}{2}(M-1)^2 + \tfrac{1}{3}(M-1)^3 \cdots.$$

When $\frac{1}{2}(M-1)^2$ is below precision of measurements, (100) $\log_e M$ is identical with percent magnification.